

*-STRUCTURES ON MODULE-ALGEBRAS

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ABSTRACT. This chapter lays out a framework for discussing *-structures on module-algebras over a Hopf *-algebra H . We define a complex conjugation functor $V \mapsto \bar{V}$, which is an involution on the module category \mathbf{HMod} , and discuss its interaction with natural constructions such as direct sums, duality, Hom, and tensor products. We define *-structures first at the level of modules. We say that V is a *-module if there is an isomorphism $*$: $\bar{V} \rightarrow V$ in \mathbf{HMod} which is involutive in an appropriate sense. Then we define *-structures on algebras in \mathbf{HMod} by requiring compatibility with multiplication. We show that a *-structure on a module lifts uniquely to the tensor algebra, and we prove that the tensor algebra has a universal mapping property for morphisms of *-modules. We also discuss inner products and adjoints in this framework. Finally, we discuss the interaction between *-structures, R -matrices, and braidings.

1. BACKGROUND AND NOTATION

1.1. Background. My motivation for writing this document was to understand what is the appropriate notion of a *-algebra in the category of modules over a Hopf *-algebra. Although this is not very deep, it is perhaps a little tricky to phrase things properly. Since *-structures are antilinear, one is forced to work with antilinear maps of complex vector spaces. This is aesthetically displeasing since one is then required to move outside the category of vector spaces and linear maps to deal with what is almost a linear phenomenon. The framework of complex conjugate linear algebra allows one to phrase everything in terms of linear maps rather than antilinear ones. It is then relatively straightforward to generalize the notions to modules (and module-algebras) over a Hopf *-algebra.

The structure of this document is as follows. In the remainder of Section 1 we set notation. In Section 2 we give the necessary background on Hopf *-algebras and discuss some properties of their module categories. In Section 3 we discuss antilinear maps, define complex conjugation as an endofunctor on the category of complex vector spaces, and show that this functor is natural with respect to many common operations on linear spaces. In Section 4 we extend the notions from Section 3 to modules over a Hopf *-algebra, and in Section 5 we examine what these notions mean for module-algebras. In Section 6 we define *-structures on modules, and in Section 7 we extend this to module-algebras. In Section 8 we treat inner products and adjoints of linear maps in our framework. Finally, in Section 9 we discuss the interaction between *-structures, R -matrices, and braidings.

1.2. Notation. All vector spaces are over \mathbb{C} . We denote the category of vector spaces over \mathbb{C} with linear maps by $\mathbb{C}\mathbf{Vect}$, and we denote the full subcategory of

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finite-dimensional vector spaces by $\mathbb{C}\mathbf{Vect}_f$. The linear dual of a vector space V will be denoted V^* . In order to avoid a proliferation of $*$'s, of which there are quite enough already, we denote the transpose (or dual map) of a linear map $T : V \rightarrow W$ by $T^{\text{tr}} : W^* \rightarrow V^*$. The vector space of all linear maps from V to W will be denoted $\text{Hom}(V, W)$. An undecorated Hom will always refer to linear maps, not module maps.

For the rest of this document, H will denote a Hopf algebra over \mathbb{C} with coproduct Δ , counit ε , and antipode S . We use the Sweedler notation

$$\Delta(a) = a_{(1)} \otimes a_{(2)}$$

for the coproduct of H , with implied summation. When we refer to H -modules, we mean left modules for the underlying algebra structure of H . We denote the category of left H -modules with H -module maps by \mathbf{HMod} , and we denote the full subcategory of finite-dimensional modules by \mathbf{HMod}_f . The vector space of all module morphisms from V to W will be denoted by $\text{Hom}_H(V, W)$. The vector space $\text{Hom}(V, W)$ of all linear maps from V to W has the structure of an H -module, but as we will see in Section 2 there are two choices for the action when H is a Hopf $*$ -algebra and one must distinguish between them.

For any category \mathcal{C} , we write $X \in \mathcal{C}$ to mean that X is an object of \mathcal{C} .

2. HOPF $*$ -ALGEBRAS AND THEIR MODULES

In this section we recall some necessary definitions and facts about Hopf $*$ -algebras. One important (and not immediately obvious) fact is that the antipode of a Hopf $*$ -algebra is invertible. This has the consequence that some linear spaces—namely the linear dual $\text{Hom}(V, \mathbb{C})$ of a module V and the space $\text{Hom}(V, W)$ of linear maps between two modules—carry two natural actions of H . We assume throughout the rest of this document that H is a Hopf $*$ -algebra (see Definition 2.2), although we note that in §2.3 and §2.4 we only require the antipode to be invertible.

Hopf $*$ -algebras are discussed, for example, in Section 1.2.7 of [KS97] and in Section 4.1.F of [CP95]. Tensor products and duals of modules, as well as the module structure on Hom -spaces, can be found in Section 4.1.C of [CP95].

Notation 2.1. The symbol \triangleright will generally indicate the action of a Hopf algebra on a module, so that $a \triangleright v$ means the action of a on v . When we consider more than one action of H on the same vector space, we will distinguish one of the actions by using the symbol \blacktriangleright rather than \triangleright .

2.1. Hopf $*$ -algebras.

Definition 2.2. A $*$ -structure on H is an antilinear map $*$: $H \rightarrow H$ such that

- (a) $*$ is involutive: $(a^*)^* = a$ for all $a \in H$.
- (b) $*$ reverses products: $(ab)^* = b^*a^*$ for all $a, b \in H$.
- (c) Δ is a $*$ -homomorphism: $\Delta(a^*) = a_{(1)}^* \otimes a_{(2)}^*$ for all $a \in H$.

If H is equipped with a $*$ -structure then we say that H is a *Hopf $*$ -algebra*.

It follows from Definition 2.2 that the counit is also a $*$ -homomorphism, i.e. that $\varepsilon(a^*) = \overline{\varepsilon(a)}$ for $a \in H$. Another consequence of the definition is that

$$(2.1) \quad * \circ S \circ * \circ S = \text{id}_H,$$

which is shown in Proposition 10, Section 1.2.7 of [KS97]. The proof proceeds by showing that $* \circ S \circ *$ is an antipode for H^{op} (which *a priori* is only a bialgebra),

and hence S is invertible with $S^{-1} = * \circ S \circ *$ by Proposition 6, Section 1.2.4 of the same reference.

Remark 2.3. The fact that the antipode is necessarily invertible shows that not every Hopf algebra over \mathbb{C} can be given a $*$ -structure. In [Tak71], a so-called free Hopf algebra $H(C)$ is constructed for any coalgebra C , and those coalgebras for which the antipode of $H(C)$ is invertible are classified. In particular, if $C = M_n(\mathbb{C})^*$ is the linear dual of a matrix algebra with $n > 1$, then the antipode of $H(C)$ is not invertible, and hence $H(C)$ cannot be equipped with a $*$ -structure.

However, there are also examples of Hopf algebras over \mathbb{C} with bijective antipode which cannot be endowed with a $*$ -structure. It is shown in Example 10, Section 1.2.7 of [KS97] that if \mathfrak{g} is a complex Lie algebra, then $*$ -structures on the universal enveloping algebra $U(\mathfrak{g})$ correspond bijectively to real forms of \mathfrak{g} . It is shown in Example 1, Section 1.7.1 of [Vin94] that the complex Lie algebra spanned by X, Y , and Z with relations

$$[X, Y] = Y, \quad [X, Z] = 2iZ, \quad [Y, Z] = 0$$

has no real form, and hence its enveloping algebra has no $*$ -structures.

2.2. Tensor products of modules. We recall that the coproduct of H allows us to form the tensor product of modules. For $V, W \in \mathbf{HMod}$ we endow $V \otimes W$ with the tensor product action

$$(2.2) \quad a \triangleright (v \otimes w) = (a_{(1)} \triangleright v) \otimes (a_{(2)} \triangleright w).$$

This gives \mathbf{HMod} the structure of a monoidal category in which the monoidal unit is the ground field \mathbb{C} , which is an H -module via the counit ε .

2.3. Duals of modules. For any vector space V we can form the dual vector space $\text{Hom}(V, \mathbb{C})$. If V is a left module for a complex algebra A , then $\text{Hom}(V, \mathbb{C})$ is naturally a right A -module, or equivalently a left A^{op} -module. For a Hopf algebra H , the antipode can be viewed as an algebra homomorphism $S : H \rightarrow H^{op}$. Hence for $V \in \mathbf{HMod}$, the linear space $\text{Hom}(V, \mathbb{C})$ becomes a left H -module via the action

$$(2.3) \quad (a \triangleright f)(v) = f(S(a) \triangleright v)$$

for all $a \in H$, $f \in \text{Hom}(V, \mathbb{C})$, and $v \in V$.

We have seen in §2.1 that the antipode S is invertible since H is a Hopf $*$ -algebra. Thus S^{-1} gives a second algebra homomorphism $H \rightarrow H^{op}$, and corresponding to this is a second left H -module structure on $\text{Hom}(V, \mathbb{C})$ given by

$$(2.4) \quad (a \blacktriangleright f)(v) = f(S^{-1}(a) \triangleright v)$$

for all $a \in H$, $f \in \text{Hom}(V, \mathbb{C})$, and $v \in V$. As this action is less well-known than the standard dual action (2.3), we briefly verify that it is an action. Linearity in a and f is clear, so we just need to check compatibility with multiplication in H . For $a, b \in H$ and $f \in \text{Hom}(V, \mathbb{C})$ we have

$$\begin{aligned} (a \blacktriangleright (b \blacktriangleright f))(v) &= (b \blacktriangleright f)(S^{-1}(a) \triangleright v) \\ &= f(S^{-1}(b) \triangleright (S^{-1}(a) \triangleright v)) \\ &= f((S^{-1}(b)S^{-1}(a)) \triangleright v) \\ &= f(S^{-1}(ab) \triangleright v) \\ &= (ab \blacktriangleright f)(v), \end{aligned}$$

so $a \blacktriangleright (b \blacktriangleright f) = ab \blacktriangleright f$.

Definition 2.4. For $V \in \mathbf{HMod}$, we define $V^* = \text{Hom}(V, \mathbb{C})$ with the action \triangleright of H given by (2.3), and we refer to this as the *left dual of V* . We define ${}^*V = \text{Hom}(V, \mathbb{C})$ with the action \blacktriangleright of H given by (2.4), and we refer to this as the *right dual of V* .

Remark 2.5. Lemma 2.6 below explains how to remember which dual is the left and which one is the right. The left dual, V^* , goes to the left of V in the evaluation pairing, while the right dual, *V , goes to the right. In both cases, the superscript $*$ is adjacent to the \otimes symbol; one can think of the $*$ as “eating” the vector.

Lemma 2.6. For $V \in \mathbf{HMod}$, we have

- (a) The evaluation map $\text{ev}_V : V^* \otimes V \rightarrow \mathbb{C}$ given by $\varphi \otimes v \mapsto \varphi(v)$ is a morphism of H -modules.
- (b) The evaluation map ${}_V \text{ev} : V \otimes {}^*V \rightarrow \mathbb{C}$ given by $v \otimes \varphi \mapsto \varphi(v)$ is a morphism of H -modules.

Proof. (a) For $f \in V^*$, $v \in V$, $a \in H$ we have

$$\begin{aligned} \text{ev}_V(a \triangleright (f \otimes v)) &= (a_{(1)} \triangleright f)(a_{(2)} \triangleright v) \\ &= f(S(a_{(1)})a_{(2)} \triangleright v) \\ &= \varepsilon(a)f(v) \\ &= a \triangleright (\text{ev}_V(f \otimes v)). \end{aligned}$$

(b) Similar to (a). □

Proposition 2.7. Let $V \in \mathbf{HMod}$, and for $v \in V$ let $\delta_v : \text{Hom}(V, \mathbb{C}) \rightarrow \mathbb{C}$ be the evaluation map $\delta_v(f) = f(v)$.

- (a) The linear map $V \rightarrow {}^*(V^*)$ given by $v \mapsto \delta_v$ is a morphism of H -modules, and it is an isomorphism for $V \in \mathbf{HMod}_f$.
- (b) The linear map $V \rightarrow ({}^*V)^*$ given by $v \mapsto \delta_v$ is a morphism of H -modules, and it is an isomorphism for $V \in \mathbf{HMod}_f$.

Proof. (a) On the one hand, for $a \in H$, $v \in V$ and $f \in V^*$ we have

$$\delta_{a \triangleright v}(f) = f(a \triangleright v),$$

while on the other hand we have

$$\begin{aligned} (a \blacktriangleright \delta_v)(f) &= \delta_v(S^{-1}(a) \triangleright f) \\ &= (S^{-1}(a) \triangleright f)(v) \\ &= f(S(S^{-1}(a)) \triangleright v) \\ &= f(a \triangleright v), \end{aligned}$$

and hence $\delta_{a \triangleright v} = a \blacktriangleright \delta_v$. If V is finite-dimensional then the dimensions of V and ${}^*(V^*)$ coincide. Since $v \mapsto \delta_v$ is clearly injective, it is also surjective, and hence is an isomorphism.

(b) Similar to (a). □

In some Hopf algebras, e.g. quasitriangular ones, the square of the antipode is an inner automorphism; see Proposition 5, Section 8.1.3 of [KS97]. In that case the left and right duals coincide, and then for finite-dimensional modules the second dual is naturally isomorphic to the original module:

Proposition 2.8. *Suppose that there is an invertible element u of H such that $S^2(a) = uau^{-1}$ for all $a \in A$. Then:*

- (a) $au \triangleright f = ua \blacktriangleright f$ for all $f \in \text{Hom}(V, \mathbb{C})$ and $a \in H$. In particular $u \triangleright f = u \blacktriangleright f$.
- (b) The linear map ${}^*V \rightarrow V^*$ given by $f \mapsto u \triangleright f$ is an isomorphism of modules whose inverse is $f \mapsto u^{-1} \triangleright f$.
- (c) The linear map $V \rightarrow V^{**}$ given by $v \mapsto \delta_{u \triangleright v}$ is a morphism of modules, and it is an isomorphism if $V \in \mathbf{HMod}_f$, where δ is as defined in Proposition 2.7.

Proof. (a) First note that for $a \in H$ we have

$$S(au) = S^{-1}(S^2(au)) = S^{-1}(ua),$$

so for any $v \in V$ we have

$$\begin{aligned} (au \triangleright f)(v) &= f(S(au) \triangleright v) \\ &= f(S^{-1}(ua) \triangleright v) \\ &= (ua \blacktriangleright f)(v), \end{aligned}$$

so $au \triangleright f = ua \blacktriangleright f$.

(b) We compute

$$u \triangleright (a \blacktriangleright f) = u \blacktriangleright (a \triangleright f) = ua \blacktriangleright f = au \triangleright f = a \triangleright (u \triangleright f),$$

using (a) for the first and third equalities. Hence $f \mapsto u \triangleright f$ is a module map ${}^*V \rightarrow V^*$.

(c) For $a \in A$ and $v \in V$ we have

$$\begin{aligned} (a \triangleright \delta_{u \triangleright v})(f) &= \delta_{u \triangleright v}(S(a) \triangleright f) \\ &= (S(a) \triangleright f)(u \triangleright v) \\ &= f(S^2(a)u \triangleright v) \\ &= f(ua \triangleright v) \\ &= \delta_{u \triangleright (a \triangleright v)}(f), \end{aligned}$$

so we see that $a \triangleright \delta_{u \triangleright v} = \delta_{u \triangleright (a \triangleright v)}$, i.e. $v \mapsto \delta_{u \triangleright v}$ is a module map. The map is injective, and hence bijective if V is finite-dimensional since the dimensions of V and V^{**} coincide. \square

Remark 2.9. The existence of left and right duals makes \mathbf{HMod}_f into a *rigid* or *autonomous* category; see, for example, Section 2.1 of [BK01] for a precise definition. If in addition S^2 is inner then part (c) of Proposition 2.8 implies that \mathbf{HMod}_f is a *pivotal* category in the sense of Definition 5.1 of [FY92].

2.4. Module structures on $\text{Hom}(V, W)$. The two actions of H on $\text{Hom}(V, \mathbb{C})$ give rise to two actions of H on the space $\text{Hom}(V, W)$ for $V, W \in \mathbf{HMod}$. The standard action is given by

$$(2.5) \quad a \triangleright T = a_{(1)}TS(a_{(2)})$$

for $T \in \text{Hom}(V, W)$ and $a \in H$. It is not difficult to check directly that (2.5) defines an action of H . However, the definition can be better motivated by viewing the space of linear maps as

$$(2.6) \quad \text{Hom}(V, W) \simeq W \otimes V^*,$$

where a simple tensor $w \otimes \varphi \in W \otimes V^*$ acts as a linear map on a vector $v \in V$ by

$$(w \otimes \varphi)(v) = w\varphi(v).$$

Note that we have written the scalar on the right here. Then it is straightforward to see that

$$\begin{aligned} [a \triangleright (w \otimes \varphi)](v) &= [(a_{(1)} \triangleright w) \otimes (a_{(2)} \triangleright \varphi)](v) \\ &= (a_{(1)} \triangleright w)(a_{(2)} \triangleright \varphi)(v) \\ &= a_{(1)} \triangleright [w\varphi(S(a_{(2)})) \triangleright v] \\ &= [a_{(1)}(w \otimes \varphi)S(a_{(2)})](v), \end{aligned}$$

which explains (2.5).

Remark 2.10. We should note that the decomposition (2.6) holds only when at least one of V and W is finite-dimensional. In general there is an injective linear map $W \otimes V^* \rightarrow \text{Hom}(V, W)$ whose range consists of all finite-rank operators. The point here is that we only use the correspondence (2.6) heuristically; the action (2.5) is well-defined no matter whether V or W is finite-dimensional. The same remarks apply to the decomposition (2.8) and the action (2.9) below as well.

In order to get the tensor product action to agree with (2.5) it was essential that we used V^* rather than *V in the decomposition (2.6). Also note that the order of the two factors in this decomposition was arranged so that evaluation of linear maps on vectors is given by

$$(2.7) \quad \text{id}_W \otimes \text{ev}_V : W \otimes V^* \otimes V \rightarrow W,$$

which itself is a morphism of modules. We prove this in general (i.e. for modules which are not necessarily finite-dimensional) in Proposition 2.15.

This tells us how we can get another canonical action of H on $\text{Hom}(V, W)$; we exchange *V for V^* and swap the order of the factors. Thus we write

$$(2.8) \quad \text{Hom}(V, W) \simeq {}^*V \otimes W,$$

with the action of a simple tensor $\varphi \otimes w \in {}^*V \otimes W$ on $v \in V$ defined by

$$(\varphi \otimes w)(v) = \varphi(v)w.$$

Remark 2.11. If we were bolder we would write v on the left of $\varphi \otimes w$ in the preceding display, but as the overwhelming convention is to write functions to the left of their arguments, we seem to be stuck with the current clunky formulation.

In any case, the decomposition (2.8) allows us to define another action of H on $\text{Hom}(V, W)$, which we denote using the symbol \blacktriangleright . For $a \in H$, $\varphi \otimes w \in {}^*V \otimes W$ and $v \in V$, the tensor product action gives

$$\begin{aligned} [a \blacktriangleright (\varphi \otimes w)](v) &= [(a_{(1)} \blacktriangleright \varphi) \otimes (a_{(2)} \triangleright w)](v) \\ &= (a_{(1)} \blacktriangleright \varphi)(v)a_{(2)} \triangleright w \\ &= a_{(2)} \triangleright [\varphi(S^{-1}(a_{(1)})) \triangleright v]w \\ &= [a_{(2)}(\varphi \otimes w)S^{-1}(a_{(1)})](v). \end{aligned}$$

Hence for $T \in \text{Hom}(V, W)$ the action \blacktriangleright of H is given by

$$(2.9) \quad a \blacktriangleright T = a_{(2)}TS^{-1}(a_{(1)}).$$

Definition 2.12. For $V, W \in \mathbf{HMod}$, we define $\text{Hom}_\ell(V, W)$ to be the linear space $\text{Hom}(V, W)$ with the action \triangleright given by (2.5), and we call this the *left Hom-space*. We define $\text{Hom}_r(V, W)$ to be the linear space $\text{Hom}(V, W)$ with the action \blacktriangleright given by (2.9), and we call this the *right Hom-space*. When $V = W$, we denote $\text{Hom}_\ell(V, V)$ and $\text{Hom}_r(V, V)$ by $\text{End}_\ell(V)$ and $\text{End}_r(V)$, respectively.

We now formalize the statements made in Remark 2.10:

Proposition 2.13. *Let $V, W \in \mathbf{HMod}$.*

- (a) *The map $W \otimes V^* \rightarrow \text{Hom}_\ell(V, W)$ given by $(w \otimes \varphi)(v) = w\varphi(v)$ for $v \in V$, $w \in W$, and $\varphi \in V^*$ is an injective module map. It is an isomorphism if V and W are finite-dimensional.*
- (b) *The map ${}^*V \otimes W \rightarrow \text{Hom}_r(V, W)$ given by $(\varphi \otimes w)(v) = \varphi(v)w$ for $v \in V$, $w \in W$, and $\varphi \in V^*$ is an injective module map. It is an isomorphism if V and W are finite-dimensional.*

Proof. The proofs that these are module maps are exactly the displayed computations immediately preceding Remark 2.10 and Definition 2.12, respectively. Injectivity is straightforward, and then these maps are isomorphisms for finite-dimensional modules just by a dimension count. \square

Remark 2.14. As with duals, there is a simple way to remember which is the left Hom-space and which is the right. The left Hom-space $\text{Hom}_\ell(V, W)$ goes to the left of V in the evaluation map, and it is constructed using the left dual of V . The right Hom-space $\text{Hom}_r(V, W)$ goes to the right of V in the evaluation map, and it is constructed using the right dual of V . This is encapsulated in:

Lemma 2.15. *For $V, W \in \mathbf{HMod}$, we have*

- (a) *The evaluation map $\text{Hom}_\ell(V, W) \otimes V \rightarrow W$ given by $T \otimes v \mapsto T(v)$ is a morphism of H -modules.*
- (b) *The evaluation map $V \otimes \text{Hom}_r(V, W) \rightarrow W$ given by $v \otimes T \mapsto T(v)$ is a morphism of H -modules.*

Proof. We do part (a) only; part (b) is similar. For finite-dimensional modules the result follows from Equation (2.7). We now give a direct proof which works for arbitrary modules. Denoting the evaluation map by ev , for $a \in H$ we have

$$\begin{aligned}
 \text{ev}(a \triangleright (T \otimes v)) &= \text{ev}((a_{(1)} \triangleright T) \otimes (a_{(2)} \triangleright v)) \\
 &= a_{(1)} \triangleright T(S(a_{(2)})a_{(3)} \triangleright v) \\
 &= a_{(1)} \triangleright T(\varepsilon(a_{(2)})v) \\
 &= a \triangleright (Tv) \\
 &= a \triangleright \text{ev}(T \otimes v). \quad \square
 \end{aligned}$$

An element v of an H -module V is called *invariant* if $a \triangleright v = \varepsilon(a)v$ for all $a \in H$. The submodule of invariant elements is denoted by V^H . For the standard action (2.5) of H on $\text{Hom}(V, W)$, it is well-known that the invariant elements are precisely the H -module maps. It is therefore natural to ask what the invariants are for the action (2.9). It turns out that they are the same:

Proposition 2.16. *For $V, W \in \mathbf{HMod}$ we have*

$$\text{Hom}_\ell(V, W)^H = \text{Hom}_H(V, W) = \text{Hom}_r(V, W)^H.$$

Proof. First suppose that T is a module map. Then for $a \in H$ we have

$$a \triangleright T = a_{(1)}TS(a_{(2)}) = a_{(1)}S(a_{(2)})T = \varepsilon(a)T,$$

and similarly we see that

$$a \blacktriangleright T = a_{(2)}TS^{-1}(a_{(1)}) = a_{(2)}S^{-1}(a_{(1)})T = \varepsilon(a)T;$$

the last equality follows since $S(a_{(2)}S^{-1}(a_{(1)})) = a_{(1)}S(a_{(2)}) = \varepsilon(a)$.

Now suppose that $T \in \text{Hom}_\ell(V, W)^H$. Using part (a) of Lemma 2.15, for $v \in V$ we have

$$\begin{aligned} a \triangleright (T(v)) &= (a_{(1)} \triangleright T)(a_{(2)} \triangleright v) \\ &= \varepsilon(a_{(1)})T(a_{(2)} \triangleright v) \\ &= T(a \triangleright v). \end{aligned}$$

Similarly, if $T \in \text{Hom}_r(V, W)^H$, we have using part (b) of Lemma 2.15 that

$$\begin{aligned} a \triangleright (T(v)) &= (a_{(2)} \blacktriangleright T)(a_{(1)} \triangleright v) \\ &= \varepsilon(a_{(2)})T(a_{(1)} \triangleright v) \\ &= T(a \triangleright v). \end{aligned}$$

□

3. ANTILINEAR MAPS AND THE CONJUGATION FUNCTOR ON $\mathbb{C}\mathbf{Vect}$

While our main goal is to describe complex conjugates of modules for Hopf $*$ -algebras, these notions also make sense for vector spaces over \mathbb{C} . Actually vector spaces are just the special case of modules over the Hopf $*$ -algebra \mathbb{C} itself, but anyway...

Here we set out the basics of complex conjugate linear algebra before moving on to study modules in Section 4. We begin with some elementary remarks on antilinear maps. Then we introduce the complex conjugation functor, our main technical tool in what follows. We show that the conjugation functor allows antilinear maps to be interpreted naturally as linear ones. Finally we discuss several natural constructions in $\mathbb{C}\mathbf{Vect}$ and examine their interactions with conjugation. We emphasize the functorial nature of the constructions throughout.

3.1. Elementary remarks on antilinear maps. When doing linear algebra over \mathbb{C} , one often encounters the notion of an antilinear map between vector spaces. A trivial example is the map $\lambda \mapsto \overline{\lambda}$ of \mathbb{C} to itself. A less trivial example is the map $T \mapsto T^*$, where T is a linear map between complex inner product spaces. Another example is a complex inner product itself, which is antilinear in the first variable.

While these maps are generally not difficult to deal with on their own, the framework of complex conjugate linear algebra allows us to understand antilinear maps while working within $\mathbb{C}\mathbf{Vect}$, at the cost of some added complexity (pun intended).

Definition 3.1. Let $V, W \in \mathbb{C}\mathbf{Vect}$. We say that a function $T : V \rightarrow W$ is *antilinear* if

$$T(\alpha u + \beta v) = \overline{\alpha}T(u) + \overline{\beta}T(v)$$

for $\alpha, \beta \in \mathbb{C}$ and $u, v \in V$. If T is bijective then we say that T is an *anti-isomorphism*; an anti-isomorphism of V with itself will be called an *anti-automorphism*. We denote the collection of antilinear maps from V to W by $\text{cHom}(V, W)$.

- Lemma 3.2.** (a) For $V, W \in \mathbb{C}\mathbf{Vect}$, the set $\mathrm{cHom}(V, W)$ is a subspace of the vector space of all functions from V to W .
 (b) The composition of two antilinear maps is linear. The composition of an antilinear map and a linear map, in either order, is antilinear.

Proof. Straightforward. \square

Remark 3.3. While the proof of Lemma 3.2 is trivial, there is some nontrivial content. While it is clear how to define addition in $\mathrm{cHom}(V, W)$, there is some choice in the definition of scalar multiplication. If we defined instead $(\lambda T)(v) = \overline{\lambda}T(v)$, this would give us a vector space structure different from that defined in Lemma 3.2. The key observation is that there is a natural vector space structure on the set of all (not necessarily linear) functions from V to W ; viewing $\mathrm{cHom}(V, W)$ inside this space allows us to choose naturally between the two options.

We would like to be able to interpret $\mathrm{cHom}(V, W)$ as a Hom -set in $\mathbb{C}\mathbf{Vect}$. *A priori* this is not possible due to the banal fact that antilinear maps are not linear. In §3.2 we introduce the complex conjugation functor, which is our main technical tool in the rest of this text. This will allow us to interpret antilinear maps naturally as linear ones.

3.2. The conjugation functor.

Definition 3.4. For $V \in \mathbb{C}\mathbf{Vect}$, we define the *complex conjugate vector space* of V or just *complex conjugate* of V to be the complex vector space \overline{V} consisting of formal symbols $c_V(v) = \overline{v}$ for $v \in V$ with addition and scalar multiplication given by

$$(3.1) \quad \overline{v} + \overline{w} = \overline{v + w}, \quad \lambda \cdot \overline{v} = \overline{\lambda v}$$

for $v, w \in V, \lambda \in \mathbb{C}$, respectively. Equivalently, we can define the operations in \overline{V} by declaring the map $c_V : V \rightarrow \overline{V}$ given by $v \mapsto \overline{v} = c_V(v)$ to be an antilinear bijection.

For a linear map $T : V \rightarrow W$, we define the *complex conjugate* of T to be the map $\overline{T} : \overline{V} \rightarrow \overline{W}$ given by $\overline{T}(\overline{v}) = \overline{T(v)}$ for $v \in V$, i.e. $\overline{T} = c_W \circ T \circ c_V^{-1}$. In other words, \overline{T} is the unique map making the diagram

$$(3.2) \quad \begin{array}{ccc} V & \xrightarrow{T} & W \\ c_V \downarrow & & \downarrow c_W \\ \overline{V} & \xrightarrow{\overline{T}} & \overline{W} \end{array}$$

commute. It follows from Lemma 3.2 that \overline{T} is linear.

Remark 3.5. The map c_V is an isomorphism of the underlying real vector spaces of V and \overline{V} . Although V and \overline{V} are isomorphic as complex vector spaces since their dimensions are the same, there is in general no natural isomorphism. The exception is the ground field \mathbb{C} , as we will see below in Lemma 3.10.

We emphasize that \overline{T} is the *unique* linear map making (3.2) commute; this follows immediately from the fact that c_V and c_W are bijective.

Note also that the map $T \mapsto \overline{T}$ is itself an antilinear map since

$$\overline{\lambda T} = c_W \circ (\lambda T) \circ c_V^{-1} = \overline{\lambda} \cdot c_W \circ T \circ c_V^{-1} = \overline{\lambda} \cdot \overline{T}.$$

Notation 3.6. We generally use the notation \bar{v} rather than the more cumbersome $c_V(v)$ except in the following two situations when confusion may arise from doing so.

The first possible source of confusion is when $V = \mathbb{C}$, where $\bar{\lambda}$ may refer either to the element $c_{\mathbb{C}}(\lambda) \in \overline{\mathbb{C}}$ or to the complex conjugate element $\bar{\lambda} \in \mathbb{C}$. We show in Lemma 3.10 that $\overline{\mathbb{C}}$ is naturally isomorphic to \mathbb{C} and that $c_{\mathbb{C}}(\lambda)$ is identified with $\bar{\lambda}$.

The second possible source of confusion occurs when discussing linear maps. We have seen that we can define the complex conjugate of a linear map $T : V \rightarrow W$; the symbol \bar{T} can then refer either to the conjugate map $\bar{T} : \bar{V} \rightarrow \bar{W}$ or to the element $c_{\text{Hom}(V,W)}(T)$ of $\overline{\text{Hom}(V,W)}$. We show in Proposition 4.10 that there is a natural way to identify \bar{T} with $c_{\text{Hom}(V,W)}(T)$.

Nevertheless, we will explicitly say what we mean whenever the notation may cause confusion.

Lemma 3.7. *Complex conjugation is a functor from the category $\mathbb{C}\mathbf{Vect}$ to itself. That is, for $V, W, X \in \mathbb{C}\mathbf{Vect}$ and linear maps $U : V \rightarrow W$ and $T : W \rightarrow X$ we have*

- (a) $\overline{\text{id}_V} = \text{id}_{\bar{V}}$.
- (b) $\overline{T \circ U} = \bar{T} \circ \bar{U}$.

Proof. (a) According to (3.2) we have

$$\begin{array}{ccc} V & \xrightarrow{\text{id}_V} & V \\ c_V \downarrow & & \downarrow c_V \\ \bar{V} & \xrightarrow{\text{id}_{\bar{V}}} & \bar{V} \end{array}$$

which proves the claim.

(b) Using (3.2) twice we have

$$\begin{array}{ccccc} V & \xrightarrow{U} & W & \xrightarrow{T} & X \\ c_V \downarrow & & \downarrow c_W & & \downarrow c_X \\ \bar{V} & \xrightarrow{\bar{U}} & \bar{W} & \xrightarrow{\bar{T}} & \bar{X} \end{array}$$

so we see that $\bar{T} \circ \bar{U} : \bar{V} \rightarrow \bar{X}$ is a linear map which fulfils the uniqueness criterion discussed in Remark 3.5, and hence $\overline{T \circ U} = \bar{T} \circ \bar{U}$. \square

The next proposition allows us to view antilinear maps as linear ones, as mentioned in Remark 3.3. In order to state the result properly, note that the functor taking a pair of vector spaces (V, W) to $\text{cHom}(V, W)$ is contravariant in V and covariant in W : given linear maps $U : V' \rightarrow V$ and $R : W \rightarrow W'$, the map $\text{cHom}(V, W) \rightarrow \text{cHom}(V', W')$ is given by $T \mapsto R \circ T \circ U$, i.e.

$$\text{cHom}(U, R) : \left(V \xrightarrow{T} W \right) \mapsto \left(V' \xrightarrow{U} V \xrightarrow{T} W \xrightarrow{R} W' \right).$$

Similarly the functor taking (V, W) to $\text{Hom}(\overline{V}, W)$ is contravariant in V and covariant in W : for linear maps $U : V' \rightarrow V$ and $R : W \rightarrow W'$, the map $\text{Hom}(\overline{V}, W) \rightarrow \text{Hom}(\overline{V'}, W')$ is given by

$$\text{Hom}(\overline{U}, R) : \left(\overline{V} \xrightarrow{T} W \right) \mapsto \left(\overline{V'} \xrightarrow{\overline{U}} \overline{V} \xrightarrow{T} W \xrightarrow{R} W' \right).$$

Proposition 3.8. *For $V, W \in \mathbb{C}\mathbf{Vect}$, the map*

$$\Psi_{VW} : \text{cHom}(V, W) \rightarrow \text{Hom}(\overline{V}, W)$$

given by $T \mapsto T \circ c_V^{-1}$ is a linear isomorphism. If T is an antilinear isomorphism then $\Psi_{VW}(T)$ is a linear isomorphism. The collection of isomorphisms (Ψ_{VW}) is a natural transformation in the sense that for linear maps $U : V' \rightarrow V$ and $R : W \rightarrow W'$, the following diagram commutes:

$$(3.3) \quad \begin{array}{ccc} \text{cHom}(V, W) & \xrightarrow{\text{cHom}(U, R)} & \text{cHom}(V', W') \\ \Psi_{VW} \downarrow & & \downarrow \Psi_{V'W'} \\ \text{Hom}(\overline{V}, W) & \xrightarrow[\text{Hom}(\overline{U}, R)]{} & \text{Hom}(\overline{V'}, W') \end{array}$$

Remark 3.9. The definition of Ψ_{VW} is best captured by a diagram. For an antilinear map $T : V \rightarrow W$, the linear map $\Psi_{VW}(T)$ is the unique map making the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ c_V \downarrow & \nearrow \Psi_{VW}(T) & \\ \overline{V} & & \end{array}$$

In fancy terms, this proposition states that for a fixed $V \in \mathbb{C}\mathbf{Vect}$, the pair $(\overline{V}, (\Psi_{V-}))$ represents the covariant functor $\text{cHom}(V, -)$.

Proof of Proposition 3.8. The image of Ψ_{VW} lands in $\text{Hom}(\overline{V}, W)$ since the composition of two antilinear maps is linear. The inverse of Ψ_{VW} is given by $T \mapsto T \circ c_V$, so indeed Ψ_{VW} is an isomorphism. Since c_V is bijective, we see that $\Psi_{VW}(T)$ is bijective if and only if T is. The naturality statement (3.3) follows from commutativity of the diagram

$$\begin{array}{ccccccc} V' & \xrightarrow{U} & V & \xrightarrow{T} & W & \xrightarrow{R} & W' \\ c_{V'} \downarrow & & c_V \downarrow & \nearrow \Psi_{VW}(T) & & & \\ \overline{V'} & \xrightarrow{\overline{U}} & \overline{V} & & & & \end{array}$$

□

3.3. Further properties of the conjugation functor. In this subsection we examine how the conjugation functor interacts with some natural constructions in $\mathbb{C}\mathbf{Vect}$. We show also that conjugation is an autoequivalence of $\mathbb{C}\mathbf{Vect}$ and that its square is naturally isomorphic to the identity functor. We begin by showing that \mathbb{C} is naturally isomorphic to its own conjugate. Our main tool in this subsection will be Proposition 3.8.

Lemma 3.10. *Let $(\lambda \mapsto \bar{\lambda}) \in \text{cHom}(\mathbb{C}, \mathbb{C})$ be the usual complex conjugation map from \mathbb{C} to itself. Then $\gamma = \Psi_{\mathbb{C}\mathbb{C}}(\lambda \mapsto \bar{\lambda}) \in \text{Hom}(\bar{\mathbb{C}}, \mathbb{C})$ is a linear isomorphism. Explicitly, γ is given by $\gamma(c_{\mathbb{C}}(\lambda)) = \bar{\lambda}$.*

Proof. Straightforward. \square

The analogue of Lemma 3.10 holds for any complex vector space with an anti-linear automorphism. Of course, any vector space has such an anti-automorphism if one fixes a basis; the point is that the anti-automorphism of \mathbb{C} is canonical.

Lemma 3.11. *The map $\sigma_V : \bar{\bar{V}} \rightarrow V$ given by $\bar{v} \mapsto v$ is a linear isomorphism. Given a linear map $T : V \rightarrow W$, the following diagram commutes:*

$$(3.4) \quad \begin{array}{ccc} \bar{\bar{V}} & \xrightarrow{\bar{T}} & \bar{\bar{W}} \\ \sigma_V \downarrow & & \downarrow \sigma_W \\ V & \xrightarrow{T} & W \end{array}$$

Proof. Note that $\sigma_V = (c_{\bar{V}} \circ c_V)^{-1} = \Psi_{\bar{V}V}(c_V^{-1})$; since both c_V and $c_{\bar{V}}$ are antilinear isomorphisms, σ_V is a linear isomorphism. Commutativity of (3.4) follows from commutativity of the diagram

$$\begin{array}{ccccc} V & \xrightarrow{c_V} & \bar{V} & \xrightarrow{c_{\bar{V}}} & \bar{\bar{V}} \\ T \downarrow & & \bar{T} \downarrow & & \bar{\bar{T}} \downarrow \\ W & \xrightarrow{c_W} & \bar{W} & \xrightarrow{c_{\bar{W}}} & \bar{\bar{W}} \end{array}$$

\square

Proposition 3.12. *The conjugation functor is an autoequivalence of $\mathbb{C}\mathbf{Vect}$ whose quasi-inverse is itself. The conjugation functor is exact.*

Proof. Lemma 3.11 shows that the square of the conjugation functor is naturally isomorphic to the identity functor of $\mathbb{C}\mathbf{Vect}$. Exactness holds because any equivalence of abelian categories is exact. \square

Lemma 3.13. *Let $V = (V_j)_{j \in J}$ be any family of vector spaces. Then the map*

$$\pi_V : \bigoplus_{j \in J} \bar{V}_j \rightarrow \bigoplus_{j \in J} \bar{\bar{V}}_j$$

given by

$$\pi_V \left(\overline{(v_j)_{j \in J}} \right) = (\bar{v}_j)_{j \in J}$$

is a linear isomorphism. If $W = (W_j)_{j \in J}$ is another family of vector spaces indexed by J and if $(T_j : V_j \rightarrow W_j)_{j \in J}$ is a family of linear maps, then the diagram

$$(3.5) \quad \begin{array}{ccc} \bigoplus_{j \in J} \bar{V}_j & \xrightarrow{\bar{T}_V} & \bigoplus_{j \in J} \bar{W}_j \\ \pi_V \downarrow & & \downarrow \pi_W \\ \bigoplus_{j \in J} \bar{\bar{V}}_j & \xrightarrow{T_{\bar{V}}} & \bigoplus_{j \in J} \bar{\bar{W}}_j \end{array}$$

commutes, where $T_V = \bigoplus_{j \in J} T_j$ and $T_{\bar{V}} = \bigoplus_{j \in J} \bar{T}_j$.

Proof. Note that $\pi_V = \Psi_{\oplus_j V_j, \oplus_j \overline{V_j}}(\oplus_j c_{V_j})$. Since $\oplus_j c_{V_j}$ is an antilinear isomorphism, it follows from Proposition 3.8 that π_V is a linear isomorphism. Commutativity of (3.5) is straightforward. \square

Remark 3.14. We may well ask about the interaction of complex conjugation with several other natural constructions in \mathbf{CVect} , namely duals, tensor products, and Hom. We defer discussion of these constructions to later sections because of additional complications which are introduced when working with H -modules rather than vector spaces.

4. COMPLEX CONJUGATION OF MODULES

In this section H is a fixed Hopf $*$ -algebra. We define complex conjugation of H -modules and define the concept of an anti-module map. We show how complex conjugation interacts with tensor products and duals of modules and we examine the complex conjugates of Hom-spaces also. We discuss how the results of Section 3 extend to H -modules.

4.1. The conjugation functor on \mathbf{HMod} . If H is a Hopf algebra over \mathbb{C} then \mathbf{HMod} is a subcategory of \mathbf{CVect} . It is therefore natural to ask whether the complex conjugation functor restricts to an endofunctor on \mathbf{HMod} . This boils down to two questions: first, whether we can define a module structure on \overline{V} for $V \in \mathbf{HMod}$; and second, whether complex conjugates of module maps are again module maps. The fact that H is a Hopf $*$ -algebra allows us to answer these questions affirmatively.

Definition 4.1. Given any module $V \in \mathbf{HMod}$, we define its *complex conjugate module* to be the complex conjugate vector space \overline{V} with action given by

$$(4.1) \quad a \triangleright \overline{v} = \overline{S(a)^* \triangleright v}, \text{ or equivalently } a \triangleright c_V(v) = c_V(S(a)^* \triangleright v)$$

for $a \in H$ and $v \in V$.

Remark 4.2. It is straightforward to check that (4.1) defines an action of H on \overline{V} ; for the sake of completeness we carry out this computation in the proof of Proposition 4.3. It is interesting to note that, except for interchanging the order of the antipode and the $*$ -operation, this is essentially the only the choice for a module structure on \overline{V} : the $*$ is necessary in order to keep the operation of a on \overline{v} linear in a , but since the $*$ reverses products, the S is required also to make \overline{V} into a left module.

If we instead switched the order of the S and the $*$ in (4.1), we would have an analogous concept. More precisely, we would define \widetilde{V} to be \overline{V} as a complex vector space, with the H -action given by $a \triangleright \widetilde{v} = \widetilde{S(a^*) \triangleright v}$ for $a \in A$ and $v \in V$. Certainly all of the following theory could be developed in that framework, and while it does not appear that there is a natural transformation which directly connects these two complex conjugation functors, there are some relationships between them.

First, we claim that $(\widetilde{V})^* \simeq {}^*\overline{V}$. Indeed, note that the underlying complex vector space of each of these modules is just $\text{Hom}(\overline{V}, \mathbb{C})$. Then for $f \in \text{Hom}(\overline{V}, \mathbb{C})$ and $v \in V$, we have

$$(a \triangleright f)(\widetilde{v}) = f(S(a) \triangleright \widetilde{v}) = f(\widetilde{S(S(a)^*) \triangleright v}) = f(\widetilde{a^* \triangleright v}),$$

while on the other hand we have

$$(a \blacktriangleright f)(\bar{v}) = f(S^{-1}(a) \triangleright \bar{v}) = f(\overline{a^* \triangleright v}),$$

so the identity map on the underlying vector space $\text{Hom}(\bar{V}, \mathbb{C})$ in fact is an isomorphism of modules $(\tilde{V})^* \simeq {}^*\bar{V}$.

Also we can consider the modules $\tilde{\bar{V}}$ and $\widetilde{\bar{V}}$. For $v \in V$ and $a \in A$, in the former module we have

$$a \triangleright \tilde{\bar{v}} = \widetilde{S(a^*) \triangleright \bar{v}} = \widetilde{S^2(a^*)^* \triangleright v} = \widetilde{S^{-2}(a) \triangleright v},$$

while in the latter we get

$$a \triangleright \widetilde{\bar{v}} = \overline{S(a)^* \triangleright \bar{v}} = \overline{S^2(a) \triangleright v}.$$

Thus the two actions are related by the automorphism S^4 of H . Hence if S^4 is an inner automorphism, the two modules are isomorphic, while if S^2 is an inner automorphism then both modules are isomorphic to V (and hence the two conjugation functors are quasi-inverse to one another). As mentioned above, if H is quasitriangular then in fact S^2 is inner. In general, however, this is not the case, although in some situations S^4 is “almost inner” in a certain sense, and this may allow one to say more about the relationship between the modules $\tilde{\bar{V}}$ and $\widetilde{\bar{V}}$. For finite-dimensional Hopf algebras the result on S^4 is due to Radford [Rad76], extending a result of Larson; for various generalizations to different classes of infinite-dimensional Hopf algebras one can see [BBT07], [BB09], and references therein.

Proposition 4.3. *Let $V, W \in \mathbf{HMod}$.*

- (a) *Equation (4.1) defines an action of H on \bar{V} , i.e. $\bar{V} \in \mathbf{HMod}$.*
- (b) *If $T : V \rightarrow W$ is a module map, then $\bar{T} : \bar{V} \rightarrow \bar{W}$ is also a module map.*
- (c) *Complex conjugation is an endofunctor of \mathbf{HMod} .*

Proof. (a) It is clear that the action is linear in H and in \bar{v} . We just check that it is compatible with multiplication in H . For $a, b \in H$ and $\bar{v} \in \bar{V}$ we have

$$\begin{aligned} a \triangleright (b \triangleright \bar{v}) &= a \triangleright \overline{S(b)^* \triangleright v} \\ &= \overline{S(a)^* \triangleright (S(b)^* \triangleright v)} \\ &= \overline{(S(a)^* S(b)^*) \triangleright v} \\ &= \overline{(S(b)S(a))^* \triangleright v} \\ &= \overline{S(ab)^* \triangleright v} \\ &= (ab) \triangleright \bar{v}. \end{aligned}$$

(b) For $v \in V$ and $a \in H$ we have

$$\begin{aligned} \bar{T}(a \triangleright \bar{v}) &= \bar{T}(\overline{S(a)^* \triangleright v}) \\ &= \overline{T(S(a)^* \triangleright v)} \\ &= \overline{S(a)^* \triangleright T(v)} \\ &= a \triangleright \overline{T(v)} \\ &= a \triangleright \bar{T}(\bar{v}) \end{aligned}$$

Thus \bar{T} is a module map.

- (c) We know already from Lemma 3.7 that complex conjugation takes identities to identities and preserves composition. By part (b) it takes module maps to module maps, so it defines a functor from \mathbf{HMod} to itself. \square

4.2. Antimodule maps. In Section 3 we defined antilinear maps and introduced complex conjugation of vector spaces as a way to turn antilinear maps into linear ones. Here we do the opposite: having defined complex conjugation of modules, we use this to motivate the definition of antimodule maps. We then show that complex conjugation of modules turns antimodule maps into module maps, as one would hope. This will prove useful later on when we show how the results of Section 3 extend to the category of H -modules.

Definition 4.4. Let $V, W \in \mathbf{HMod}$. We say that a function $T : V \rightarrow W$ is an *antimodule map* if T is antilinear and satisfies

$$T(a \triangleright v) = S(a)^* \triangleright T(v)$$

for all $a \in H$ and $v \in V$. We denote the collection of antimodule maps from V to W by $\text{cHom}_H(V, W)$.

Of course, the prototype for the definition of an antimodule map is $c_V : V \rightarrow \overline{V}$. The analogue of Lemma 3.2 (b) holds:

Lemma 4.5. *The composition of two antimodule maps is a module map. The composition of an antimodule map and a module map, in either order, is an antimodule map.*

Proof. Straightforward. \square

Proposition 4.6. *Let $V, W \in \mathbf{HMod}$. Then the linear isomorphism Ψ_{VW} from Proposition 3.8 restricts to an isomorphism*

$$\Psi_{VW} : \text{cHom}_H(V, W) \rightarrow \text{Hom}_H(\overline{V}, W).$$

The analogue of the naturality statement (3.3) holds with respect to module maps $S : V' \rightarrow V$ and $R : W \rightarrow W'$.

Proof. Since c_V is an antimodule map, Lemma 4.5 implies that Ψ_{VW} carries antimodule maps to module maps. The inverse of Ψ_{VW} is given by $T \mapsto T \circ c_V$. The naturality statement is immediate from (3.3). \square

Proposition 4.7. *Let V and W be objects in \mathbf{HMod} and let $\mathbf{V} = (V_j)_{j \in J}$ be a family of objects in \mathbf{HMod} .*

- (a) *The map $\gamma : \overline{\mathbb{C}} \rightarrow \mathbb{C}$ from Lemma 3.10 is a module isomorphism.*
- (b) *The map $\sigma_V : \overline{\overline{V}} \rightarrow V$ from Lemma 3.11 is a module isomorphism.*
- (c) *The map $\pi_V : \overline{\bigoplus_{j \in J} V_j} \rightarrow \bigoplus_{j \in J} \overline{V_j}$ from Lemma 3.13 is a module isomorphism.*

Proof. We have already shown that γ, σ_V , and π_V are linear isomorphisms in Section 3, so it is only left to show that they are morphisms of modules. Since these maps are all constructed from antilinear maps using the natural isomorphisms Ψ , our strategy is to show that these antilinear maps are actually *antimodule* maps; then the result will follow from Proposition 4.6.

- (a) For $a \in H$ and $\lambda \in \mathbb{C}$ we have

$$\overline{a \triangleright \lambda} = \overline{\varepsilon(a)\lambda} = \overline{\varepsilon(S(a))} \cdot \overline{\lambda} = \varepsilon(S(a)^*) \overline{\lambda} = S(a)^* \triangleright \overline{\lambda},$$

so $\lambda \mapsto \overline{\lambda}$ is an antimodule map. Hence $\gamma = \Psi_{\mathbb{C}\mathbb{C}}(\lambda \mapsto \overline{\lambda})$ is a module map.

- (b) Since c_V is an antimodule map, $\sigma_V = \Psi_{\overline{V}V}(c_V^{-1})$ is a module map.
(c) Each c_{V_j} is an antimodule map, so $\oplus_j c_{V_j}$ is as well. Hence $\pi_V = \Psi_{\oplus_j V_j, \oplus_j \overline{V}_j}(\oplus_j c_{V_j})$ is a module map. \square

4.3. Conjugation of tensor products. In this subsection we show that complex conjugation reverses the order of tensor products of modules.

Proposition 4.8. *Given any modules $V, W \in \mathbf{HMod}$, the map $\rho_{VW} : \overline{V} \otimes \overline{W} \rightarrow \overline{W} \otimes \overline{V}$ given by $\overline{v} \otimes \overline{w} \mapsto \overline{w} \otimes \overline{v}$ is an isomorphism of H -modules. Given module maps $S : W \rightarrow Y$ and $T : X \rightarrow Z$, the diagram*

$$(4.2) \quad \begin{array}{ccc} \overline{W} \otimes \overline{X} & \xrightarrow{\overline{S} \otimes \overline{T}} & \overline{Y} \otimes \overline{Z} \\ \rho_{WX} \downarrow & & \downarrow \rho_{YZ} \\ \overline{X} \otimes \overline{W} & \xrightarrow[\overline{T} \otimes \overline{S}]{} & \overline{Z} \otimes \overline{Y} \end{array}$$

commutes. Furthermore, for any U, V, W in \mathbf{HMod} , the diagram

$$(4.3) \quad \begin{array}{ccc} \overline{U} \otimes \overline{V} \otimes \overline{W} & \xrightarrow{\rho_{U,V} \otimes \text{id}_W} & \overline{V} \otimes \overline{W} \otimes \overline{U} \\ \rho_{U \otimes V, W} \downarrow & & \downarrow \rho_{VW} \otimes \text{id}_U \\ \overline{W} \otimes \overline{U} \otimes \overline{V} & \xrightarrow[\text{id}_{\overline{W}} \otimes \rho_{UV}]{} & \overline{W} \otimes \overline{V} \otimes \overline{U} \end{array}$$

commutes.

Proof. Our goal is to show that $\eta : v \otimes w \mapsto \overline{w} \otimes \overline{v}$ is an antimodule map; then it will follow from Proposition 4.6 that $\rho_{VW} = \Psi_{V \otimes W, \overline{W} \otimes \overline{V}}(\eta)$ is a module map. Noting that $\Delta(S(a)^*) = S(a_{(2)})^* \otimes S(a_{(1)})^*$, we have

$$\begin{aligned} \eta(a \triangleright v \otimes w) &= \eta(a_{(1)} \triangleright v \otimes a_{(2)} \triangleright w) \\ &= \overline{a_{(2)}} \triangleright \overline{w} \otimes \overline{a_{(1)}} \triangleright \overline{v} \\ &= (S(a_{(2)})^* \triangleright \overline{w}) \otimes (S(a_{(1)})^* \triangleright \overline{v}) \\ &= S(a)^* \triangleright \eta(v \otimes w), \end{aligned}$$

so η is an antimodule map, as claimed. It is clear that ρ_{VW} is a linear isomorphism and that (4.2) and (4.3) commute. \square

4.4. Conjugation of duals. It is natural to ask whether (left or right) dualization commutes with complex conjugation of modules. It turns out that the situation is slightly subtler than that. In fact, complex conjugation intertwines left and right duals, i.e. $\overline{V^*} \simeq {}^* \overline{V}$. This can be seen as follows. Consider the evaluation pairing $\text{ev}_V : V^* \otimes V \rightarrow \mathbb{C}$. Taking the complex conjugate of this map, composing with the isomorphism $\gamma : \overline{\mathbb{C}} \rightarrow \mathbb{C}$, and precomposing with the isomorphism $\rho_{V^*V}^{-1}$ of $\overline{V} \otimes \overline{V^*}$ with $\overline{V^*} \otimes \overline{V}$ gives

$$\gamma \circ \overline{\text{ev}_V} \circ \rho_{V^*V}^{-1} : \overline{V} \otimes \overline{V^*} \rightarrow \mathbb{C};$$

thus $\overline{V^*}$ plays the role of the right dual of \overline{V} . We formalize this as follows:

Proposition 4.9. *Let $V, W \in \mathbf{HMod}$.*

- (a) *The map $b_V : V^* \rightarrow {}^* \overline{V}$ given by*

$$b_V(f)(\overline{v}) = \overline{f(v)}$$

is a bijective antimodule map.

(b) The map $\beta_V : \overline{V^*} \rightarrow {}^* \overline{V}$ given by

$$\beta_V = \Psi_{V^*, {}^* \overline{V}}(b_V)$$

is an isomorphism of modules.

(c) For any module map $T : V \rightarrow W$, the following diagram commutes:

$$(4.4) \quad \begin{array}{ccc} \overline{W^*} & \xrightarrow{\overline{T^{\text{tr}}}} & \overline{V^*} \\ \beta_W \downarrow & & \downarrow \beta_V \\ {}^* \overline{W} & \xrightarrow{{}^* \overline{T^{\text{tr}}}} & {}^* \overline{V} \end{array}$$

(d) The collection of maps (β_V) is a natural equivalence of the (contravariant) endofunctors of **HMod** given by left dual followed by conjugation, and conjugation followed by right dual, respectively.

Proof. (a) Note first that $b_V(f) = (\lambda \mapsto \overline{\lambda}) \circ f \circ c_V^{-1}$, where $\lambda \mapsto \overline{\lambda}$ is the complex conjugation map of \mathbb{C} to itself. This shows both that $b_V(f)$ is a linear functional on \overline{V} and that b_V itself is an antilinear map. The inverse of b_V is given by $g \mapsto (\lambda \mapsto \overline{\lambda}) \circ g \circ c_V$, so b_V is bijective. Thus we just need to show that b_V is an antimodule map. For $f \in V^*$, $v \in V$, and $a \in H$ we have

$$\begin{aligned} (a \blacktriangleright b_V(f))(\overline{v}) &= b_V(f)(S^{-1}(a) \triangleright \overline{v}) \\ &= b_V(f)(\overline{a^* \triangleright v}) \\ &= \overline{f(a^* \triangleright v)} \\ &= \overline{(S^{-1}(a^*) \triangleright f)(v)} \\ &= b_V(S(a)^* \triangleright f)(\overline{v}), \end{aligned}$$

so $a \blacktriangleright b_V(f) = b_V(S(a)^* \triangleright f)$. Hence b_V is an antimodule map.

(b) Immediate from (a) together with Proposition 4.6.

(c) Straightforward.

(d) Immediate from (b) and (c). \square

4.5. Conjugation of Hom's. The fact that conjugation switches left and right duals implies also that conjugation switches left and right Hom-spaces. Heuristically, we have

$$\overline{\text{Hom}_\ell(V, W)} \simeq \overline{W \otimes V^*} \simeq \overline{V^*} \otimes \overline{W} \simeq {}^* \overline{V} \otimes \overline{W} \simeq \text{Hom}_r(\overline{V}, \overline{W}),$$

where the first isomorphism comes from the decomposition (2.6), the second from Proposition 4.8, the third from Proposition 4.9, and the last from (2.8). As we noted in Remark 2.10, these tensor product decompositions of the Hom-spaces are valid only when at least one of V and W is finite-dimensional. However, the following results are valid for all modules since they refer only to actions on Hom-spaces and not to the tensor product decompositions.

Proposition 4.10. *Let $V, W \in \mathbf{HMod}$.*

(a) The map $b_V : \text{Hom}_\ell(V, W) \rightarrow \text{Hom}_r(\overline{V}, \overline{W})$ given by

$$b_V(T) = \overline{T}$$

is a bijective antimodule map.

(b) The map $\beta_V : \overline{\text{Hom}_\ell(V, W)} \rightarrow \text{Hom}_r(\overline{V}, \overline{W})$ given by

$$\beta_V = \Psi_{\text{Hom}_\ell(V, W), \text{Hom}_r(\overline{V}, \overline{W})}(b_V)$$

is an isomorphism of modules. On elements β_V is given by

$$\beta_V(c_{\text{Hom}_\ell(V, W)}(T)) = \overline{T}.$$

Proof. (a) Note first that $b_V(T) = c_W \circ T \circ c_V^{-1}$. This shows both that $b_V(T)$ is a linear map $\overline{V} \rightarrow \overline{W}$ and that b_V itself is an antimodule map. The inverse of b_V is given by $U \mapsto c_W^{-1} \circ U \circ c_V$, so b_V is bijective. Thus we just need to show that b_V is an antimodule map. For $T \in \text{Hom}(V, W)$, $v \in V$, and $a \in H$ we have

$$\begin{aligned} (a \blacktriangleright \overline{T})(\overline{v}) &= a_{(2)} \triangleright (\overline{T}(S^{-1}(a_{(1)}) \triangleright \overline{v})) \\ &= a_{(2)} \triangleright (\overline{T(a_{(1)}^* \triangleright v)}) \\ &= a_{(2)} \triangleright \overline{T(a_{(1)}^* \triangleright v)} \\ &= \overline{S(a_{(2)})^* \triangleright T(a_{(1)}^* \triangleright v)} \\ &= \overline{(S(a)^* \triangleright T)(v)} \\ &= \overline{(S(a)^* \triangleright T)}(\overline{v}), \end{aligned}$$

so we see that

$$a \blacktriangleright b_V(T) = a \blacktriangleright \overline{T} = \overline{S(a)^* \triangleright T} = b_V(S(a)^* \triangleright T)$$

and hence b_V is an antimodule map.

(b) Follows immediately from (a) together with Proposition 4.6. \square

Remark 4.11. Part (b) of Proposition 4.10 realizes the identification of the linear map \overline{T} with the abstract element $c_{\text{Hom}(V, W)}(T)$ that we promised in the discussion immediately preceding Lemma 3.7.

5. COMPLEX CONJUGATION OF ALGEBRAS

In this section we look at algebras in the category **HMod** and we show that the complex conjugate of an algebra is again an algebra. We examine tensor algebras of modules in this light and show that for a module V , both $\overline{T(V)}$ and $T(\overline{V})$ satisfy a universal property with respect to lifting of *antimodule* maps from V into algebras in **HMod**.

For an introduction to module-algebras over a Hopf algebra one can see Section 1.3.3 of [KS97] or Section 4.1.C of [CP95]; a much deeper treatment can be found in Chapter 4 of [Mon93].

5.1. The complex conjugate of an algebra. Recall that an H -module algebra is a \mathbb{C} -algebra (A, m, u) , where $A \in \mathbf{HMod}$ and $m : A \otimes A \rightarrow A$ and $u : \mathbb{C} \rightarrow A$ are morphisms in **HMod**. We will generally just say that A is an algebra in **HMod** and refer to the maps m and u explicitly only when necessary; we write the product as $m(x \otimes y) = xy$ and the unit as $u(1_{\mathbb{C}}) = 1_A$. In terms of elements, the fact that m and u are module maps means that for $a \in H$ and $x, y \in A$ we have

$$(5.1) \quad a \triangleright (xy) = (a_{(1)} \triangleright x)(a_{(2)} \triangleright y) \quad \text{and} \quad a \triangleright 1_A = \varepsilon(a)1_A.$$

If A and B are algebras in **HMod**, then we say that $f : A \rightarrow B$ is a *morphism of module-algebras* or a *module-algebra morphism* if f is simultaneously a module map and an algebra homomorphism.

Proposition 5.1. (a) If (A, m, u) is an algebra in **HMod** then $(\overline{A}, m_{\overline{A}}, u_{\overline{A}})$ is an algebra in **HMod** with the structure maps given by

$$u_{\overline{A}} = \overline{u} \circ \gamma^{-1}, \quad m_{\overline{A}} = \overline{m} \circ \rho_{AA}^{-1},$$

where $\gamma : \overline{\mathbb{C}} \rightarrow \mathbb{C}$ and $\rho_{AA} : \overline{A} \otimes \overline{A} \rightarrow \overline{A} \otimes \overline{A}$ are the isomorphisms from Proposition 4.7 (a) and Proposition 4.8, respectively.

- (b) If $f : A \rightarrow B$ is a morphism of module-algebras, then $\overline{f} : \overline{A} \rightarrow \overline{B}$ is also a morphism of module-algebras.
- (c) Complex conjugation is an endofunctor of the category of H -module algebras with module-algebra morphisms.

Remark 5.2. Unwinding the definition of the multiplication in \overline{A} , we see that for $a, b \in A$ we have

$$\overline{a} \cdot \overline{b} = m_{\overline{A}}(\overline{a} \otimes \overline{b}) = \overline{m}(b \otimes a) = \overline{m(b \otimes a)} = \overline{ba};$$

this implies that

$$\overline{1_A} \cdot \overline{a} = \overline{a \cdot 1_A} = \overline{a}$$

and similarly $\overline{a} \cdot \overline{1_A} = \overline{a}$. To summarize, in the conjugate algebra \overline{A} we have

$$(5.2) \quad \overline{a} \cdot \overline{b} = \overline{ba} \quad \text{and} \quad 1_{\overline{A}} = \overline{1_A}.$$

Proof of Proposition 5.1. (a) By Proposition 4.3, both \overline{m} and \overline{u} are module maps.

Then $m_{\overline{A}}$ and $u_{\overline{A}}$ are both module maps, as they are compositions of module maps. We just need to verify that $m_{\overline{A}}$ and $u_{\overline{A}}$ satisfy the associativity and unit laws. But this is clear from (5.2).

- (b) We already know that \overline{f} is a module morphism by Proposition 4.3, so we just need to check that it is an algebra homomorphism. For $a, b \in A$ we have

$$\begin{aligned} \overline{f}(\overline{a} \cdot \overline{b}) &= \overline{f(ba)} \\ &= \overline{f(ba)} \\ &= \overline{f(b)f(a)} \\ &= \overline{f(a)} \cdot \overline{f(b)} \\ &= \overline{f(a)} \cdot \overline{f(b)}, \end{aligned}$$

so \overline{f} is multiplicative. We have also

$$\overline{f}(1_{\overline{A}}) = \overline{f}(\overline{1_A}) = \overline{f(1_A)} = \overline{1_B} = 1_{\overline{B}},$$

so \overline{f} is unital, and hence is an algebra homomorphism.

- (c) Follows immediately from (a) and (b) since we already know that conjugation respects identity maps and composition. \square

We now investigate some properties of morphisms of module-algebras and the corresponding conjugate-linear notions. For this we need the following:

Definition 5.3. If A and B are algebras in **HMod** then we say that a function $f : A \rightarrow B$ is an *antimodule-algebra morphism* if f is simultaneously an antimodule map, $f(ab) = f(b)f(a)$ for all $a, b \in A$, and $f(1_A) = 1_B$.

Lemma 5.4. Let A and B be algebras in **HMod**.

- (a) The composition of two antimodule-algebra morphisms is a module-algebra morphism. The composition of an antimodule-algebra morphism and a module-algebra morphism, in either order, is an antimodule-algebra morphism.
- (b) The map $c_A : A \rightarrow \overline{A}$ is an antimodule-algebra morphism.
- (c) If $f : A \rightarrow B$ is an antimodule map, then f is an antimodule-algebra morphism if and only if $\Psi_{AB}(f) : \overline{A} \rightarrow B$ is a module-algebra morphism.

Proof. (a) Straightforward.

(b) We have $c_A(1_A) = \overline{1_A} = 1_{\overline{A}}$, and for $a, b \in A$ we have

$$c_A(ab) = \overline{ab} = \overline{b} \overline{a} = c_A(b) c_A(a).$$

(c) Since $\Psi_{AB}(f) = f \circ c_A^{-1}$, the result follows immediately from parts (b) and (c). \square

Corollary 5.5. *The morphism $\gamma : \overline{\mathbb{C}} \rightarrow \mathbb{C}$ from Lemma 3.10 is an isomorphism of module-algebras.*

Proof. Since $\gamma = \Psi_{\mathbb{C}\mathbb{C}}(\lambda \mapsto \overline{\lambda})$, this follows from Lemma 5.4 (c) and the observation that $\lambda \mapsto \overline{\lambda}$ is an antimodule-algebra morphism. \square

Corollary 5.6. *If A is an algebra in \mathbf{HMod} then the map $\sigma_A : \overline{\overline{A}} \rightarrow A$ introduced in Lemma 3.11 is an isomorphism of module-algebras.*

Proof. We know that σ_A is a linear isomorphism from Lemma 3.11 and that it is a module map by Proposition 4.7. Since $\sigma_A = c_A^{-1} \circ c_{\overline{A}}^{-1}$, we have that σ_A is a morphism of module-algebras from parts (a) and (b) of Lemma 5.4. \square

5.2. The complex conjugate of the tensor algebra. For V in \mathbf{HMod} , the tensor algebra of V is defined to be

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n},$$

where by definition $V^{\otimes 0} = \mathbb{C}$ (with H -action given by the counit), and all of the tensor products are taken over \mathbb{C} . We denote by ι_V the canonical injection of V into $T(V)$, and note that ι_V is a module map. We denote homogeneous elements of $T(V)$ by $v_1 \dots v_n$ rather than $v_1 \otimes \dots \otimes v_n$ in order to save space.

Our goal is to show that there is a natural isomorphism of module-algebras between $\overline{T(V)}$ and $T(\overline{V})$. In order to do so, we begin with a discussion on the universal mapping property of the tensor algebra. Although the following result is well-known, we include the formal statement and proof for the sake of completeness.

Lemma 5.7. *Let $V \in \mathbf{HMod}$ and let $T(V)$ be defined as above.*

- (a) $T(V)$ is an algebra in \mathbf{HMod} with multiplication given by concatenation of tensors and the unit $\mathbb{C} \rightarrow T(V)$ given by the isomorphism $\mathbb{C} \simeq V^{\otimes 0}$.
- (b) For any algebra A in \mathbf{HMod} and any module map $f : V \rightarrow A$, there is a unique morphism of module-algebras $\tilde{f} : T(V) \rightarrow A$ such that $\tilde{f} \circ \iota_V = f$.

Proof. (a) The fact that $T(V)$ is an associative algebra over \mathbb{C} is straightforward. The crux of the matter is to show that the multiplication and unit are maps of H -modules. For this proof only we denote the multiplication map by $m :$

$T(V) \otimes T(V) \rightarrow T(V)$. For $v_1, \dots, v_m, w_1, \dots, w_n \in V$, denote $v = v_1 \dots v_m$ and $w = w_1 \dots w_n$. Then for $a \in H$ we have

$$\begin{aligned} m(a \triangleright (v \otimes w)) &= m((a_{(1)} \triangleright v) \otimes (a_{(2)} \triangleright w)) \\ &= (a_{(1)} \triangleright v_1) \dots (a_{(m)} \triangleright v_m) (a_{(m+1)} \triangleright w_1) \dots (a_{(m+n)} \triangleright w_n) \\ &= a \triangleright (v_1 \dots v_m w_1 \dots w_n) \\ &= a \triangleright m(v \otimes w), \end{aligned}$$

so we see that all we really used was coassociativity of the comultiplication. The fact that the unit is a module map is immediate, as the module action on $\mathbb{C} \simeq V^{\otimes 0}$ is just given by the counit of H .

- (b) Fix $n \geq 0$. If $n = 0$, define $f^0 : V^{\otimes 0} \rightarrow A$ by $f^0(1) = 1_A$. If $n > 0$, the map $\prod_{j=1}^n V \rightarrow A$ given by

$$(v_1, \dots, v_n) \mapsto f(v_1) \dots f(v_n)$$

is \mathbb{C} -multilinear and hence induces a unique map $f^n : V^{\otimes n} \rightarrow A$ given by

$$f^n(v_1 \dots v_n) = f(v_1) \dots f(v_n);$$

recall that we are omitting \otimes signs inside $T(V)$. We claim that f^n is a map of H -modules. For $a \in A$, we have

$$\begin{aligned} f^n(a \triangleright (v_1 \dots v_n)) &= f^n((a_{(1)} \triangleright v_1) \dots (a_{(n)} \triangleright v_n)) \\ &= f(a_{(1)} \triangleright v_1) \dots f(a_{(n)} \triangleright v_n) \\ &= (a_{(1)} \triangleright f(v_1)) \dots (a_{(n)} \triangleright f(v_n)) \\ &= a \triangleright (f(v_1) \dots f(v_n)) \\ &= a \triangleright f^n(v_1 \dots v_n), \end{aligned}$$

which verifies the claim. Finally, we define \tilde{f} to be the direct sum of the maps f^n for $n \geq 0$. It is immediate from the definition that \tilde{f} is an algebra homomorphism. It is also a map of H -modules since it is a direct sum of such maps. Clearly $\tilde{f} \circ \iota_V = f$, and \tilde{f} is unique with this property since $\iota(V)$ generates $T(V)$ as an algebra. \square

We will use similar ideas to show that $T(\overline{V})$ and $\overline{T(V)}$ are isomorphic; both of these algebras are equipped with antimodule maps from V and they satisfy a universal mapping property with respect to algebras with antimodule maps coming from V . Then the usual abstract nonsense will show that there is a unique isomorphism between $T(\overline{V})$ and $\overline{T(V)}$.

We define maps $\theta_V : V \rightarrow T(\overline{V})$ and $\vartheta_V : V \rightarrow \overline{T(V)}$ by

$$(5.3) \quad \theta_V = \iota_{\overline{V}} \circ c_V \quad \text{and} \quad \vartheta_V = c_{T(V)} \circ \iota_V,$$

where ι_V and $\iota_{\overline{V}}$ are the embeddings of V and \overline{V} into $T(V)$ and $T(\overline{V})$, respectively, as defined in the discussion at the beginning of the subsection. Both θ_V and ϑ_V are antimodule maps by Lemma 4.5.

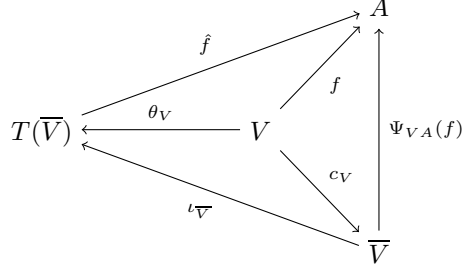
Proposition 5.8. *Let $V \in \mathbf{HMod}$ and let θ_V and ϑ_V be as in (5.3). Let A be an algebra in \mathbf{HMod} and let $f : V \rightarrow A$ be an antimodule map.*

- (a) *There is a unique module-algebra morphism $\hat{f} : T(\overline{V}) \rightarrow A$ such that $\hat{f} \circ \theta_V = f$.*
(b) *There is a unique module-algebra morphism $\check{f} : \overline{T(V)} \rightarrow A$ such that $\check{f} \circ \vartheta_V = f$.*

Proof. (a) By Proposition 4.6, $\Psi_{VA}(f) : \overline{V} \rightarrow A$ is a module map. Then the universal property of the tensor algebra implies that there is a unique module-algebra morphism $\hat{f} : T(\overline{V}) \rightarrow A$ such that $\hat{f} \circ \iota_{\overline{V}} = \Psi_{VA}(f)$. Then we have

$$\hat{f} \circ \theta_V = \hat{f} \circ \iota_{\overline{V}} \circ c_V = \Psi_{VA}(f) \circ c_V = f.$$

This is encapsulated in the commutativity of the diagram



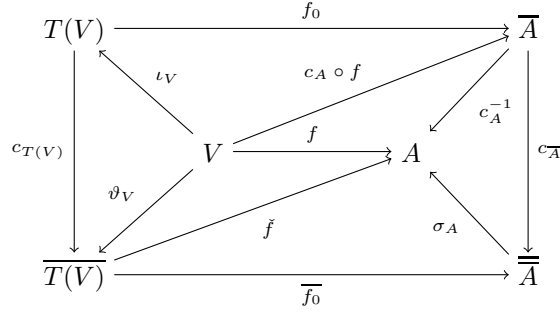
The lower left triangle commutes by definition of θ_V , the right-hand triangle commutes by definition of $\Psi_{VA}(f)$, and the large triangle commutes by the universal property of the tensor algebra. Hence the upper left triangle commutes as well, and this shows that \hat{f} has the desired property.

To see that \hat{f} is unique, note that $\theta_V(V)$ generates $T(\overline{V})$ as an algebra, and the equation $\hat{f} \circ \theta_V = f$ uniquely specifies \hat{f} on the generators. Since \hat{f} is an algebra homomorphism, this implies that it is uniquely specified on the algebra $T(\overline{V})$.

- (b) By Lemma 4.5, the function $c_A \circ f : V \rightarrow \overline{A}$ is a module map. The universal property of $T(V)$ then gives a unique extension to a module-algebra morphism $f_0 : T(V) \rightarrow \overline{A}$ such that $f_0 \circ \iota_V = c_A \circ f$. Then $\overline{f_0} : \overline{T(V)} \rightarrow \overline{\overline{A}}$ is a morphism of module-algebras by Proposition 5.1 (b). We define $\check{f} : \overline{T(V)} \rightarrow A$ by $\check{f} = \sigma_A \circ \overline{f_0}$. Since σ_A is a module-algebra morphism by Corollary 5.6, then \check{f} is a module-algebra morphism by Lemma 5.4 (a). We have

$$\begin{aligned} \check{f} \circ \vartheta_V &= \sigma_A \circ \overline{f_0} \circ c_{T(V)} \circ \iota_V \\ &= \sigma_A \circ c_{\overline{A}} \circ f_0 \circ \iota_V \\ &= c_A^{-1} \circ c_A \circ f \\ &= f. \end{aligned}$$

This is encapsulated in the commutativity of the diagram



The large rectangle commutes by (3.2). For the five triangles starting on the left-hand side of the diagram and going clockwise, the reasons they commute are, respectively: by definition of ϑ_V ; by the universal property of $T(V)$ (i.e. since f_0 lifts $c_A \circ f$); trivially; by definition of σ_A ; and by definition of \check{f} . Then the final interior triangle commutes because everything else does, and this shows that \check{f} has the desired property.

The proof of uniqueness from part (a) carries over here almost verbatim. \square

Since $T(\overline{V})$ and $\overline{T(V)}$ satisfy the same universal property, they are isomorphic:

Corollary 5.9. *There is a unique isomorphism of module-algebras $\kappa_V : T(\overline{V}) \rightarrow \overline{T(V)}$ such that $\kappa_V(\overline{v}) = \overline{v}$ for all $v \in V$. On simple tensors κ_V is given by*

$$(5.4) \quad \kappa_V(\overline{v_1} \dots \overline{v_n}) = \overline{v_n \dots v_1},$$

and hence the restriction of κ_V to $\overline{V}^{\otimes n}$ gives an isomorphism $\overline{V}^{\otimes n} \simeq \overline{V^{\otimes n}}$.

Proof. By Proposition 5.8 (a) applied to the antimodule map $\vartheta_V : V \rightarrow \overline{T(V)}$, there is a unique map of module-algebras $\kappa_V : T(\overline{V}) \rightarrow \overline{T(V)}$ such that $\kappa_V \circ \theta_V = \vartheta_V$, i.e. such that $\kappa_V(\overline{v}) = \overline{v}$.

Applying part (b) of Proposition 5.8 to the antimodule map $\theta_V : V \rightarrow T(\overline{V})$ gives a unique map of module-algebras $\psi_V : \overline{T(V)} \rightarrow T(\overline{V})$ such that $\psi_V \circ \vartheta_V = \theta_V$.

According to part (v) of Proposition 5.8, there is a unique morphism of module-algebras $\vartheta_V : \overline{T(V)} \rightarrow \overline{T(V)}$ such that $\vartheta_V \circ \vartheta_V = \vartheta_V$; it is clear that $\vartheta_V = \text{id}_{\overline{T(V)}}$. But on the other hand, we have

$$(\kappa_V \circ \psi_V) \circ \vartheta_V = \kappa_V \circ \theta_V = \vartheta_V,$$

so we must have $\kappa_V \circ \psi_V = \text{id}_{\overline{T(V)}}$ by uniqueness. Similarly we have $\psi_V \circ \kappa_V = \text{id}_{T(\overline{V})}$, so both maps are isomorphisms of module-algebras.

Finally, the formula (5.4) follows immediately from the description of multiplication in the conjugate of an algebra given in Remark 5.2. \square

We now use Proposition 5.8 to show that $T(V)$ has a universal property allowing us to lift antimodule morphisms from V to antimodule-algebra morphisms:

Proposition 5.10. *Let $V \in \mathbf{HMod}$, let A be an algebra in \mathbf{HMod} , and let $f : V \rightarrow A$ be an antimodule map. Then there is a unique antimodule-algebra morphism $f^\sharp : T(V) \rightarrow A$ such that $f^\sharp \circ \iota_V = f$.*

Proof. By part (b) of Proposition 5.8 implies that there is a unique homomorphism of module-algebras $\check{f} : \overline{T(V)} \rightarrow A$ such that $\check{f} \circ \vartheta_V = f$. Then define f^\sharp by $f^\sharp = \check{f} \circ c_{T(V)}$, i.e. so that the top right triangle of the diagram

$$\begin{array}{ccc} T(V) & \xrightarrow{c_{T(V)}} & \overline{T(V)} \\ \uparrow \iota_V & \searrow f^\sharp & \downarrow \check{f} \\ V & \xrightarrow{f} & A \end{array}$$

commutes; recall that $\vartheta_V = c_{T(V)} \circ \iota_V$. The square commutes by the defining property of \check{f} , so the lower left triangle commutes as well, i.e. we have $f^\sharp \circ \iota_V = f$.

Then f^\sharp is an antimodule-algebra morphism by Lemma 5.4, and it is unique since it is uniquely determined on the generators $\iota_V(V)$ of $T(V)$. \square

6. *-STRUCTURES ON MODULES

We now turn to the main theme of this chapter, which is *-structures. Although our goal is to formulate the correct notion of a *-algebra in the category **HMod**, it is useful to consider modules first and then extend to algebras afterward. In Section 6.1 we begin by defining *-modules and *-morphisms. We show in Proposition 6.8 that *-modules are the “fixed points up to homotopy” of the conjugation functor in the sense that the conjugation functor is naturally isomorphic to the identity functor when restricted to the subcategory of **HMod** consisting of *-modules. We then show that modules of the form $\overline{V} \otimes V$ and $V \otimes \overline{V}$ carry natural *-structures; these *-modules will be used later in Section 8 to formulate the notion of an inner product in our framework. We also discuss *-submodules and quotients.

6.1. The category \mathbf{HMod}^* . The usual definition of a *-structure on, say, a complex algebra A is an antilinear map $*$: $A \rightarrow A$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$. Simply omitting the last condition then gives a reasonable definition of a *-structure on a vector space. Translating this into our framework of complex conjugate modules using the map Ψ_{VV} leads to the following definition:

Definition 6.1. We say that a module $V \in \mathbf{HMod}$ is a **-module* if V is equipped with a module map $*$: $\overline{V} \rightarrow V$, such that $* \circ \overline{*} = \sigma_{VV} : \overline{\overline{V}} \rightarrow V$. We refer to the map $*$ as a **-structure on V* .

If $W \in \mathbf{HMod}$ is also a *-module, then we say that a linear map $T : V \rightarrow W$ is a **-morphism* or a **-map* if $T \circ * = * \circ \overline{T}$, i.e. if the following diagram commutes:

$$(6.1) \quad \begin{array}{ccc} \overline{V} & \xrightarrow{*} & V \\ \overline{T} \downarrow & & \downarrow T \\ \overline{W} & \xrightarrow{*} & W \end{array}$$

If T is also a module map then we may refer to it as a **-module morphism* or **-module map*. If $v \in V$ and $\overline{v}^* = v$ then we will say that v is a *self-adjoint element* or just that v is *self-adjoint*.

Remark 6.2. It might seem more natural for a *-structure on V to be a map originating from V rather than from \overline{V} . However, since a *-structure in the usual formulation is an antilinear map from V to V , Proposition 4.6 tells us that Definition 6.1 is appropriate.

It follows immediately from the requirement $* \circ \overline{*} = \sigma_V$ that the map $*$ itself must be an isomorphism of modules, hence we do not require it in the definition.

Lemma 6.3. (a) For any *-module V , id_V is a morphism of *-modules.
(b) The composition of two *-module morphisms is a *-module morphism.

Proof. Straightforward. \square

Definition 6.4. The preceding lemma tells us that the collection of all *-modules in **HMod** together with all *-module maps forms a subcategory of **HMod**. We denote this category by **HMod**^{*}.

Notation 6.5. We will mostly not need to work with more than one *-structure on any given module at a time, so we generally employ the notation $*$ for all *-structures we consider. If necessary we will decorate the $*$'s with labels.

If $V \in \mathbf{HMod}^*$, we write \overline{v}^* rather than $*(\overline{v})$. In this notation, the requirement that $* \circ \overline{*} = \sigma_V$ means that $((\overline{v})^*)^* = v$ for all $v \in V$. It is quite cumbersome to carry around the $\overline{*}$ in the superscript, and henceforth we will just write $*$ for the morphism $\overline{V} \rightarrow V$ as well as for its complex conjugate $\overline{\overline{V}} \rightarrow \overline{V}$. Thus the condition of involutivity becomes the slightly more palatable $((\overline{v})^*)^* = v$; omitting parentheses gives the almost satisfactory $\overline{v}^{**} = v$.

We now show that our definition of a *-structure is equivalent to the usual definition in terms of an involutive antilinear (antimodule) map:

Proposition 6.6. *Let $V, W \in \mathbf{HMod}$.*

- (a) *If $\dagger : v \mapsto v^\dagger$ is an antimodule map $V \rightarrow V$, then $* = \Psi_{VV}(\dagger) : \overline{V} \rightarrow V$ is a *-structure in the sense of Definition 6.1 if and only if $(v^\dagger)^\dagger = v$ for all $v \in V$.*
- (b) *Let $\dagger : V \rightarrow V$ and $\dagger : W \rightarrow W$ be involutive antimodule maps with corresponding *-structures as in (a). Then a linear map $T : V \rightarrow W$ is a *-map if and only if $T(v^\dagger) = (Tv)^\dagger$ for all $v \in V$.*

Proof. (a) From Proposition 4.6 we know that $*$ is a module map. For $v \in V$ we have

$$\overline{v}^* = \Psi_{VV}(\dagger)(\overline{v}) = v^\dagger,$$

so $\overline{\overline{v}^*} = \overline{v^\dagger}$. Hence

$$\overline{\overline{v}^{**}} = (\overline{v^\dagger})^* = (v^\dagger)^\dagger,$$

so we see that $\overline{\overline{v}^{**}} = \sigma_V(\overline{v}) = v$ if and only if $(v^\dagger)^\dagger = v$.

- (b) On the one hand we have

$$T(\overline{v}^*) = T(v^\dagger),$$

while on the other hand we have

$$(\overline{T(\overline{v})})^* = (\overline{Tv})^* = (Tv)^\dagger,$$

so we see that the diagram (6.1) commutes if and only if $T(v^\dagger) = (Tv)^\dagger$. \square

Remark 6.7. The upshot of Proposition 6.6 is that we can work with involutive antimodule maps rather than directly with Definition 6.1. This is convenient, especially in the proofs below, where this approach allows us to avoid the complication of carrying $\overline{}$'s around all over the place.

Proposition 6.8. *Let $V, W \in \mathbf{HMod}^*$.*

- (a) *The module \overline{V} is a *-module with the *-structure $\overline{*} : \overline{\overline{V}} \rightarrow \overline{V}$.*
- (b) *If $T : V \rightarrow W$ is a morphism of *-modules then $\overline{T} : \overline{V} \rightarrow \overline{W}$ is a morphism of *-modules with respect to the *-structures defined in (a).*
- (c) *Complex conjugation is an endofunctor of \mathbf{HMod}^* , and moreover the morphisms $* : \overline{V} \rightarrow V$ give a natural isomorphism of the complex conjugation functor with the identity functor of \mathbf{HMod}^* .*

Proof. (a) The complex conjugate morphism $\overline{*} : \overline{\overline{V}} \rightarrow \overline{V}$ is a module morphism by Proposition 4.3. Then we have (trivially and unenlighteningly)

$$\overline{*} \circ \overline{*} = \overline{* \circ *} = \overline{\sigma_V} = \sigma_{\overline{V}},$$

so $\bar{*}$ is a $*$ -structure on \bar{V} .

- (b) Follows immediately from taking the complex conjugate of the diagram (6.1).
- (c) Since each morphism $* : \bar{V} \rightarrow V$ is an isomorphism, the statement is exactly the commutativity of the diagram (6.1). \square

6.2. Building up the category \mathbf{HMod}^* . Although we have defined the category \mathbf{HMod}^* , we have not exhibited any modules in it. In this subsection we show that \mathbb{C} is a $*$ -module and that we can in fact construct a large class of $*$ -modules without knowing anything about the specific Hopf $*$ -algebra H that we are working with. We also discuss direct sums and tensor products of $*$ -modules.

Lemma 6.9. *The map $\gamma : \bar{\mathbb{C}} \rightarrow \mathbb{C}$ is a $*$ -structure on \mathbb{C} .*

Proof. Follows immediately from Proposition 6.6. \square

Proposition 6.10. *Let $V \in \mathbf{HMod}$.*

- (a) *The module $V^e = \bar{V} \otimes V$ is a $*$ -module with $*$ -structure given by*

$$(\overline{v \otimes w})^* = \bar{w} \otimes v$$

for $v, w \in V$.

- (b) *The module ${}^eV = V \otimes \bar{V}$ is a $*$ -module with $*$ -structure given by*

$$(\overline{v \otimes \bar{w}})^* = w \otimes \bar{v}$$

for $v, w \in V$.

Proof. We show the proof for (a) only; (b) is similar. Let us define a map $\dagger : V^e \rightarrow V^e$ by

$$(\bar{v} \otimes w)^\dagger = \bar{w} \otimes v.$$

It is clear that $(x^\dagger)^\dagger = x$ for all $x \in V^e$. Moreover we claim that \dagger is an antimodule map. For $v, w \in V$ and $a \in H$ we have

$$\begin{aligned} (a \triangleright (\bar{v} \otimes w))^\dagger &= [(a_{(1)} \triangleright \bar{v}) \otimes (a_{(2)} \triangleright w)]^\dagger \\ &= [\overline{(S(a_{(1)}))^* \triangleright v}] \otimes (a_{(2)} \triangleright w)]^\dagger \\ &= (\overline{a_{(2)} \triangleright \bar{w}}) \otimes (S(a_{(1)})^* \triangleright v) \\ &= (S(a_{(2)})^* \triangleright \bar{w}) \otimes (S(a_{(1)})^* \triangleright v) \\ &= S(a)^* \triangleright (\bar{w} \otimes v) \\ &= S(a)^* \triangleright (\bar{v} \otimes w)^\dagger, \end{aligned}$$

which verifies the claim. Then Proposition 6.6 implies that $* = \Psi_{V^e V^e}(\dagger)$ is a $*$ -structure. \square

Proposition 6.11. *For a family $(V_j)_{j \in J}$ in \mathbf{HMod}^* with $*$ -structures $* : \bar{V}_j \rightarrow V_j$, the direct sum $\bigoplus_{j \in J} V_j$ is a $*$ -module with the $*$ -structure given by*

$$(6.2) \quad (\overline{(v_j)_{j \in J}})^* = (\bar{v}_j^*)_{j \in J}.$$

Proof. For each $j \in J$, define a map $\dagger : V_j \rightarrow V_j$ by $\dagger = \Psi_{V_j V_j}^{-1}(*_j)$. Each \dagger is an antimodule map by Proposition 4.6, and $(v_j^\dagger)^\dagger = v_j$ for all $v_j \in V_j$ by Proposition 6.6.

Then we define $\dagger : \bigoplus_{j \in J} V_j \rightarrow \bigoplus_{j \in J} V_j$ to be the direct sum of the maps $\dagger : V_j \rightarrow V_j$ and observe that \dagger is an involutive antimodule map since all of its direct summands are such. Finally, the map (6.2) is given by

$$* = \Psi_{\bigoplus_j V_j, \bigoplus_j V_j}(\dagger),$$

so it is a *-structure by Proposition 6.6. \square

Proposition 6.12. *For $V \in \mathbf{HMod}^*$ and n a positive integer, the module $V^{\otimes n}$ is a *-module with the *-structure given by*

$$(6.3) \quad (\overline{v_1 \otimes \cdots \otimes v_n})^* = \overline{v_n}^* \otimes \cdots \otimes \overline{v_1}^*.$$

Proof. Let $\dagger = \Psi_V^{-1}(*) : V \rightarrow V$ be the involutive antimodule map corresponding to the given *-structure, and then define another map $\dagger : V^{\otimes n} \rightarrow V^{\otimes n}$ by

$$(v_1 \otimes \cdots \otimes v_n)^\dagger = v_n^\dagger \otimes \cdots \otimes v_1^\dagger.$$

This map is involutive by construction, and we claim that it is an antimodule map as well. For $a \in H$ we have

$$\begin{aligned} [a \triangleright (v_1 \otimes \cdots \otimes v_n)]^\dagger &= [(a_{(1)} \triangleright v_1) \otimes \cdots \otimes (a_{(n)} \triangleright v_n)]^\dagger \\ &= (a_{(n)} \triangleright v_n)^\dagger \otimes \cdots \otimes (a_{(1)} \triangleright v_1)^\dagger \\ &= (S(a_{(n)})^* \triangleright v_n^\dagger) \otimes \cdots \otimes (S(a_{(1)})^* \triangleright v_1^\dagger) \\ &= S(a)^* \triangleright (v_n^\dagger \otimes \cdots \otimes v_1^\dagger) \\ &= S(a)^* \triangleright (v_1 \otimes \cdots \otimes v_n)^\dagger, \end{aligned}$$

so the claim is verified. Finally, the map (6.3) is given by $* = \Psi_{V^{\otimes n}, V^{\otimes n}}(\dagger)$, so it is a *-structure by Proposition 6.6. \square

Remark 6.13. It seems that in general there is no way to define a *-structure on the tensor product of two arbitrary *-modules V and W . In order to do so we would need a module map $* : \overline{V \otimes W} \rightarrow V \otimes W$; since $\overline{V \otimes W} \simeq \overline{W} \otimes \overline{V}$, however, the tensor product of the *-structures gives us a map $* : \overline{V \otimes W} \rightarrow W \otimes V$. In some situations there is a braiding on the category \mathbf{HMod} (or \mathbf{HMod}_f), and if there is some compatibility between the braidings and the *-structures, then more can be said. We will revisit this issue later in Section 9.

It seems also that there is no natural way to define a *-structure on the dual of a *-module. Indeed, if $* : \overline{V} \rightarrow V$ is a *-structure, then the dual map is $*^{\text{tr}} : V^* \rightarrow \overline{V}^* \cong \overline{V^*}$, and taking the complex conjugate of this gives a map $\overline{V^*} \rightarrow V^*$, which is not of the correct form to be a *-structure.

6.3. *-submodules and quotients. We now prove the unsurprising fact that the quotient of a *-module V by a submodule W inherits a *-structure when W is stable under the *-operation, and we show also that the kernel and image of a *-module morphism are *-submodules.

Definition 6.14. Let $V \in \mathbf{HMod}^*$ and let $W \subseteq V$ be a submodule. We say that W is a *-submodule if $\overline{W}^* \subseteq W$. (Note that involutivity of the *-structure on V implies that $\overline{W}^* = W$ if W is a *-submodule.)

Lemma 6.15. *Let $V \in \mathbf{HMod}^*$ and let W be a $*$ -submodule. Then there is a unique $*$ -structure on the quotient V/W such that the quotient map $q : V \rightarrow V/W$ is a morphism of $*$ -modules, i.e. so that the following diagram commutes:*

$$(6.4) \quad \begin{array}{ccc} \overline{V} & \xrightarrow{*} & V \\ \overline{q} \downarrow & & \downarrow q \\ \overline{V/W} & \xrightarrow[*]{} & V/W \end{array}$$

Proof. Let $\dagger : V \rightarrow V$ be the involutive antimodule map corresponding to $*$ by Proposition 6.6. The fact that W is a $*$ -submodule implies that $W^\dagger \subseteq W$. Then $(v + W)^\dagger = v^\dagger + W$ is a well-defined involutive antimodule map on V/W , so Proposition 6.6 (a) implies that the corresponding module map is a $*$ -structure. By definition we have $q(v)^\dagger = q(v^\dagger)$, so q is a $*$ -map by part (b) of Proposition 6.6. \square

Lemma 6.16. *Let $V, W \in \mathbf{HMod}^*$ and let $T : V \rightarrow W$ be a $*$ -module map.*

- (a) *$\ker(T)$ is a $*$ -submodule of V .*
- (b) *$\operatorname{im}(T)$ is a $*$ -submodule of W .*

Proof. (a) For $v \in \ker(T)$, we have

$$T(\overline{v}^*) = (\overline{T(v)})^* = (\overline{0})^* = 0,$$

so $\overline{v}^* \in \ker(T)$, and hence $\ker(T)$ is a $*$ -submodule of V .

- (b) For $w \in \operatorname{im}(T)$, write $w = T(v)$ for some $v \in V$. Then we have

$$\overline{w}^* = (\overline{T(v)})^* = T(\overline{v}^*) \in \operatorname{im}(T),$$

so $\operatorname{im}(T)$ is a $*$ -submodule of W . \square

7. $*$ -STRUCTURES ON ALGEBRAS

In this section we extend to algebras our discussion of $*$ -structures on modules from Section 6. We define $*$ -algebras and their morphisms and show how to construct $*$ -algebra structures from involutive antimodule-algebra maps. We discuss $*$ -ideals and quotients, and give a criterion for when an ideal generated by a submodule of a $*$ -algebra is a $*$ -ideal. We show that $*$ -structures on a module can be lifted to a $*$ -structure on its tensor algebra, and further that the tensor algebra has a universal mapping property for $*$ -module morphisms. We also make some observations about $*$ -structures on algebras presented by generators and relations.

7.1. Algebras in \mathbf{HMod}^* .

Definition 7.1. Suppose that A is an algebra in \mathbf{HMod} and that A is equipped with a $*$ -structure in the sense of Definition 6.1. Then we say that A is a $*$ -module algebra, or just a $*$ -algebra in \mathbf{HMod} if $*$: $\overline{A} \rightarrow A$ is an algebra homomorphism, and we refer to $*$ as a $*$ -algebra structure on A .

If A and B are $*$ -algebras in \mathbf{HMod} , then a $*$ -homomorphism is an algebra homomorphism $f : A \rightarrow B$ which is also a map of $*$ -modules.

Proposition 7.2. *Suppose that A is an algebra in \mathbf{HMod} and $\dagger : A \rightarrow A$ is an antimodule map. Then $*$ = $\Psi_{AA}(\dagger)$ is a $*$ -algebra structure on A if and only if \dagger is an antimodule-algebra homomorphism satisfying $(a^\dagger)^\dagger = a$ for all $a \in A$.*

Proof. Proposition 6.6 (a) implies that $*$ is a $*$ -module structure on A if and only if \dagger is an involutive antimodule map. Then Lemma 5.4 (c) implies that $*$ is a module-algebra morphism if and only if \dagger is an antimodule-algebra morphism. \square

7.2. *-ideals and quotients. Here we show that, as one might expect, the quotient of a $*$ -algebra by an ideal inherits a $*$ -structure exactly when the ideal is a $*$ -algebra. This will be useful in §7.3 when we discuss $*$ -structures on algebras defined by generators and relations.

Definition 7.3. An ideal (two-sided) I in a $*$ -algebra A in **HMod** is called a $*$ -ideal if $(\bar{I})^* \subseteq I$. (Note that this implies that $(\bar{I})^* = I$ because of involutivity.)

Lemma 7.4. Let A be a $*$ -algebra in **HMod** and suppose that the ideal I is a $*$ -submodule of A in the sense of Definition 6.14. With respect to the $*$ -structure on A/I defined in Lemma 6.15, the quotient map $q : A \rightarrow A/I$ is a morphism of $*$ -module algebras.

Proof. The quotient map is a map of module-algebras by construction, and it is a $*$ -map by Lemma 6.15. \square

Lemma 7.5. Let A be a $*$ -algebra in **HMod** and let $W \subseteq A$ be a $*$ -submodule. Then the two-sided ideal J_W of A generated by W is a $*$ -submodule of A , and hence is a $*$ -ideal.

Proof. The elements of J_W are finite sums of terms of the form awb , where $a, b \in A$ and $w \in W$. For $x \in H$ we have

$$x \triangleright (awb) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright w)(x_{(3)} \triangleright b);$$

since W is a submodule, the middle term $x_{(2)} \triangleright w$ is in W , so $x \triangleright (awb)$ is in J_W . Hence J_W is a submodule.

To see that J_W is a $*$ -ideal, we need to show that $(\overline{awb})^* \in J_W$ for all $a, b \in A$ and $w \in W$. We have

$$(\overline{awb})^* = (\overline{b\bar{w}\bar{a}})^* = \bar{b}^* \bar{w}^* \bar{a}^*;$$

since W is a $*$ -submodule, the middle term \bar{w}^* is in W , so $(\overline{awb})^*$ is in J_W . Thus J_W is a $*$ -ideal in A . \square

7.3. *-structures on $T(V)$. Here we show that a $*$ -structure on a module V extends uniquely to the tensor algebra $T(V)$, and we show that $T(V)$ has a universal mapping property for $*$ -algebras equipped with module maps coming from V .

Proposition 7.6. Let $V \in \mathbf{HMod}^*$ and let $T(V)$ be the tensor algebra of V . There is a unique $*$ -module algebra structure on $T(V)$ such that the inclusion map $\iota_V : V \rightarrow T(V)$ is a $*$ -module morphism, i.e. such that the following diagram commutes:

$$(7.1) \quad \begin{array}{ccc} \bar{V} & \xrightarrow{*} & V \\ \iota_V \downarrow & & \downarrow \iota_V \\ \overline{T(V)} & \xrightarrow[*]{} & T(V) \end{array}$$

For an element $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$, we have

$$(7.2) \quad (\overline{v_1 \otimes \cdots \otimes v_n})^* = \bar{v}_n^* \otimes \cdots \otimes \bar{v}_1^*.$$

Hence the $*$ -structure on $T(V)$ coincides with the direct sum of the $*$ -structures on the modules $V^{\otimes n}$ coming from Proposition 6.12.

Proof. Let $\dagger : V \rightarrow V$ be the involutive antimodule map corresponding to the given $*$ -structure on V . Now $\iota_V \circ \dagger$ is an antimodule map $V \rightarrow T(V)$, so by Proposition 5.10 it extends uniquely to an antimodule-algebra morphism, which we also denote $\dagger : T(V) \rightarrow T(V)$. Note that $\dagger \circ \dagger$ is a module-algebra endomorphism of $T(V)$ which is the identity on the generators, so it must be the identity map. Hence Proposition 6.6 (a) implies that $*$ = $\Psi_{T(V)T(V)}(\dagger)$ is a $*$ -structure on $T(V)$.

The fact that ι_V is a $*$ -map follows from commutativity of the diagram

$$\begin{array}{ccc}
 \overline{V} & \xrightarrow{\overline{\iota_V}} & \overline{T(V)} \\
 \downarrow * & \swarrow c_V^{-1} & \nwarrow c_{T(V)}^{-1} \\
 & V & \xrightarrow{\iota_V} T(V) \\
 \downarrow * & \swarrow \dagger & \nwarrow \dagger \\
 V & \xrightarrow{\iota_V} & T(V)
 \end{array}$$

The upper trapezoid commutes by definition of the complex conjugate of ι_V . The triangle on the right commutes by the definition of $*$. The lower trapezoid commutes by Proposition 5.10. The triangle on the left commutes by definition of \dagger . Thus the large rectangle commutes as well, which gives (7.1).

The $*$ -structure on $T(V)$ is unique with this property since $T(V)$ is generated as an algebra by $\iota_V(V)$, and (7.1) uniquely specifies $*$ on the generators.

The equation (7.2) follows from the fact that $*$ is an algebra map together with the description (5.2) of multiplication in the complex conjugate of an algebra. We see that (7.2) is identical to (6.3), so this $*$ -structure on $T(V)$ is indeed the direct sum of the star structures coming from Proposition 6.12. \square

Remark 7.7. We now briefly discuss $*$ -structures on algebras given by generators and relations. This means the following: we take a module V (the generators) and a submodule $W \subseteq T(V)$ (the relations), and form the quotient algebra $A = T(V)/J_W$, where J_W is the 2-sided ideal generated by W as in Lemma 7.5. If V is a $*$ -module, then $T(V)$ is a $*$ -algebra, and we would like to know when this $*$ -structure descends to A . By Lemma 7.4, it is sufficient for J_W to be a $*$ -ideal. Then by Lemma 7.5, it is sufficient that W is a $*$ -submodule of $T(V)$. According to Lemma 6.16, this happens, for instance, when W is the kernel or image of a morphism of $*$ -modules. While these observations are more or less trivial in the abstract, they may be useful later on in specific circumstances when checking relations by hand is complicated.

Proposition 7.8. *Let $V \in \mathbf{HMod}^*$, and endow $T(V)$ with the corresponding $*$ -structure as described in Proposition 7.6. Let A be a $*$ -algebra in \mathbf{HMod} and let $f : V \rightarrow A$ be a $*$ -module map. Then the morphism $\tilde{f} : T(V) \rightarrow A$ from Lemma 5.7 is in fact a morphism of $*$ -algebras.*

Proof. We know from Lemma 5.7 that \tilde{f} is a morphism of module-algebras, so we only need to show that \tilde{f} respects the $*$ -structures on $T(V)$ and A . We claim that

the following diagram commutes:

$$\begin{array}{ccccc}
 \overline{T(V)} & \xrightarrow{\quad * \quad} & T(V) & & \\
 \downarrow \bar{f} & \swarrow \bar{\iota}_V & \nearrow \iota_V & & \downarrow \tilde{f} \\
 & \overline{V} & \xrightarrow{\quad * \quad} & V & \\
 & \searrow \bar{f} & \nearrow f & & \\
 \overline{A} & \xrightarrow{\quad * \quad} & A & &
 \end{array}$$

Indeed, the upper trapezoid commutes by Proposition 7.6, the triangle on the right commutes by Lemma 5.7, the lower trapezoid commutes since f is a $*$ -map, and the triangle on the left is just the complex conjugate of the triangle on the right. Thus the large rectangle commutes also, which means that \tilde{f} is a $*$ -map. \square

8. INNER PRODUCTS AND ADJOINTS

We now turn to a discussion of inner products. In the traditional formulation, an inner product on a complex vector space V is a map $(,) : V \times V \rightarrow \mathbb{C}$ which is antilinear in the first variable, linear in the second variable, satisfies $(v, w) = \overline{(w, v)}$ for $v, w \in V$, and which is positive definite. This definition, like that of a $*$ -structure, does not lend itself easily to a module-theoretic approach because of the antilinearity. Inner products allow one to introduce the notion of adjoint linear transformations; this too is an antilinear concept. In this section we develop these notions in our framework.

Although so far we have discussed $*$ -structures only on modules, not on arbitrary complex vector spaces, the discussion extends to complex vector spaces just by taking $H = \mathbb{C}$ with $*$ -structure given by complex conjugation. In particular, we can speak of $*$ -vector spaces and morphisms of $*$ -vector spaces.

8.1. Inner products. We now formulate the definition of an inner product in the language of complex conjugate vector spaces and modules:

Definition 8.1. For $V \in \mathbb{C}\mathbf{Vect}$ we define an *inner product* on V to be a linear map $\langle , \rangle : V^e = \overline{V} \otimes V \rightarrow \mathbb{C}$ such that

- (a) With respect to the $*$ -structures on V^e and \mathbb{C} constructed in Proposition 6.10 and Lemma 3.10, respectively, \langle , \rangle is a morphism of $*$ -vector spaces.
- (b) $\langle \bar{v}, v \rangle \geq 0$ for all $v \in V$, and $\langle \bar{v}, v \rangle = 0$ only if $v = 0$.

If in addition V is an H -module and if \langle , \rangle is a module map, then we say that \langle , \rangle is an inner product in \mathbf{HMod} ; in this case we will call V together with this inner product a *Hermitian module*. If V and W are Hermitian modules with inner products \langle , \rangle_V and \langle , \rangle_W , respectively, then we say that a module map $T : V \rightarrow W$ is an *isometry* if $\langle \overline{T}u, Tv \rangle_W = \langle \bar{u}, v \rangle_V$ for all $u, v \in V$. We say that T is *unitary* if T is an invertible isometry.

Remark 8.2. Untangling the definitions shows that requiring \langle , \rangle to be a $*$ -map means just that $\langle \bar{v}, w \rangle = \overline{\langle \bar{w}, v \rangle}$ for all $v, w \in V$.

As we have done in previous sections, we could make the Hermitian modules into a category where the morphisms are the isometries. This is somewhat restrictive,

as the isometry condition forces all morphisms to be injective, and for our purposes we don't need to consider this category separately, so we omit the definition.

Notation 8.3. As we did in Definition 8.1, we will frequently abuse notation by writing $h(w, x)$ rather than $h(w \otimes x)$ when $h : W \otimes X \rightarrow Y$ is a linear map. This is nothing more than the canonical identification of bilinear maps $W \times X \rightarrow Y$ with linear ones $W \otimes X \rightarrow Y$, and we will make this identification without further comment from now on.

We now show that Definition 8.1 is equivalent to the usual definition of an inner product, and we formulate the appropriate notion of H -invariance for the traditional incarnation $(,) : V \times V \rightarrow \mathbb{C}$ of the inner product in the case when V is an H -module:

Proposition 8.4. *Let $V \in \mathbb{C}\mathbf{Vect}$ and let $\langle , \rangle : \overline{V} \otimes V \rightarrow \mathbb{C}$ be a linear map. Define a function $(,) : V \times V \rightarrow \mathbb{C}$ by $(v, w) = \langle \overline{v}, w \rangle$ for $v, w \in V$. Then*

- (a) *\langle , \rangle is an inner product in the sense of Definition 8.1 if and only if $(,)$ satisfies the usual definition of an inner product given at the beginning of §8.*
- (b) *If $V \in H\mathbf{Mod}$, then \langle , \rangle is a module map if and only if $(,)$ is H -invariant in the sense that $(a \triangleright v, w) = (v, a^* \triangleright w)$ for all $v, w \in V$ and $a \in H$.*

Proof. (a) Since $v \mapsto \overline{v}$ is antilinear, it is clear that (v, w) is antilinear in v , and linearity in w is also clear. The formulations of positive-definiteness for \langle , \rangle and $(,)$ are also clearly equivalent. We claim now that \langle , \rangle is a $*$ -map, i.e. that the diagram

$$\begin{array}{ccc} \overline{\overline{V} \otimes V} & \xrightarrow{\overline{\langle , \rangle}} & \overline{\mathbb{C}} \\ * \downarrow & & \downarrow \gamma \\ \overline{V} \otimes V & \xrightarrow{\langle , \rangle} & \mathbb{C} \end{array}$$

commutes, if and only if $(,)$ satisfies conjugate-symmetry. For $v, w \in V$, chasing $\overline{\overline{v} \otimes w}$ right and then down in the diagram gives

$$\gamma(c_{\mathbb{C}}(\langle \overline{v}, w \rangle)) = \overline{\langle \overline{v}, w \rangle} \stackrel{\text{def}}{=} \overline{(v, w)}.$$

On the other hand, chasing $\overline{\overline{v} \otimes w}$ down and then right gives

$$\langle \overline{\overline{v} \otimes w^*} \rangle = \langle \overline{w}, v \rangle \stackrel{\text{def}}{=} (w, v),$$

so our claim is verified.

- (b) First, observe that for H -modules W and X , a linear map $h : W \otimes X \rightarrow \mathbb{C}$ (i.e. a bilinear form) is a module map if and only if $h(a \triangleright w, x) = h(w, S(a) \triangleright x)$ for all $w \in W$ and $x \in X$. Then for $v, w \in V$ and $a \in H$ we have

$$(a \triangleright v, w) = \langle \overline{a \triangleright v}, w \rangle = \langle S(a)^* \triangleright \overline{v}, w \rangle,$$

while on the other hand we have

$$(v, a^* \triangleright w) = \langle \overline{v}, a^* \triangleright w \rangle.$$

Thus $(a \triangleright v, w) = (v, a^* \triangleright w)$ for all $a \in H$ if and only if $\langle S(a)^* \triangleright \overline{v}, w \rangle = \langle \overline{v}, a^* \triangleright w \rangle$ for all $a \in H$. Replacing a with $S(a)^*$ and using (2.1), the latter equation becomes $\langle a \triangleright \overline{v}, w \rangle = \langle \overline{v}, S(a) \triangleright w \rangle$. Then our first observation concludes the proof. \square

Proposition 8.5. *Suppose that $V \in \mathbf{HMod}$ is a Hermitian module. Then $\mu_V : \overline{V} \rightarrow V^*$ given by $\mu_V(\overline{v}) = \langle \overline{v}, - \rangle$ is an injective module map, and μ_V is an isomorphism if V is finite-dimensional. If $W \in \mathbf{HMod}$ is another Hermitian module and the module map $T : V \rightarrow W$ is an isometry, then the following diagram commutes:*

$$(8.1) \quad \begin{array}{ccc} \overline{V} & \xrightarrow{\overline{T}} & \overline{W} \\ \mu_V \downarrow & & \downarrow \mu_W \\ V^* & \xleftarrow{T^{\text{tr}}} & W^* \end{array}$$

Remark 8.6. We caution the reader that μ_V is not intrinsic to V ; it is only defined relative to a fixed inner product.

Proof of Proposition 8.5. Since $\mu_V = \Psi_{VV^*}(v \mapsto \langle \overline{v}, - \rangle)$, we need to show that $v \mapsto \langle \overline{v}, - \rangle$ is an antimodule map. For $a \in H$ and $v, w \in V$ we have

$$\begin{aligned} \langle \overline{a \triangleright v}, w \rangle &= \langle S(a)^* \triangleright \overline{v}, w \rangle \\ &= \langle \overline{v}, S(S(a)^*) \triangleright w \rangle \\ &= (S(a)^* \triangleright \langle \overline{v}, - \rangle)(w), \end{aligned}$$

where for the second equality we used the fact that $\langle \cdot, \cdot \rangle$ is a module map, and for the third we used the definition (2.3) of the H -action on V^* . Hence μ_V is a module map by Proposition 4.6. Injectivity follows immediately from the fact that $\langle \cdot, \cdot \rangle$ is positive-definite, and then μ_V must be an isomorphism when V is finite-dimensional because the dimensions of \overline{V} and V^* coincide.

Finally, for any $u, v \in V$, we have

$$\begin{aligned} [T^{\text{tr}} \mu_W \overline{T}](v) &= [\mu_W(\overline{T}u)](Tv) \\ &= \langle \overline{T}u, Tv \rangle \\ &= \langle \overline{u}, v \rangle \\ &= [\mu_V(\overline{u})](v), \end{aligned}$$

so $T^{\text{tr}} \mu_W \overline{T} = \mu_V$, and hence (8.1) commutes. \square

Remark 8.7. Many authors define an inner product to be linear in the first variable and antilinear in the second. While this makes no difference for vector spaces, it matters for modules: if we were to translate this alternate definition into our framework, an inner product would become a positive-definite $*$ -module map $\langle \cdot, \cdot \rangle : V \otimes \overline{V} \rightarrow \mathbb{C}$. With this definition, Proposition 8.5 would instead give an isomorphism $\overline{V} \simeq {}^*V$. All of the subsequent theory would carry over, *mutatis mutandis*, but we omit it here for the sake of brevity.

8.2. Some remarks on positivity. We have seen in the proof of Proposition 8.4 that the conjugate-symmetry condition $(v, w) = \overline{(w, v)}$ is encapsulated in the requirement that $\langle \cdot, \cdot \rangle$ respects the $*$ -structures on $V^e = \overline{V} \otimes V$ and \mathbb{C} . Thus we can express conjugate-symmetry entirely as a condition on maps in the category rather than as a condition involving elements. This raises the question of whether the positivity criterion for an inner product can also be phrased in such a manner. It appears that the answer is no, although this may be a failure of imagination rather than an insight into necessity. One way to avoid this question entirely would be

to include the notion of “positive cone” as part of the data when discussing inner products. We will begin with an example and then give a general definition.

The fact that \langle , \rangle is a $*$ -map implies that any self-adjoint element of V^e is mapped to \mathbb{R} (the self-adjoint part of \mathbb{C}). Positivity is a further restriction: it requires that the particular self-adjoint elements in V^e of the form $\bar{v} \otimes v$ are mapped into the non-negative real numbers \mathbb{R}^+ . If we denote by $(V^e)^+$ the \mathbb{R}^+ -span of the vectors $\bar{v} \otimes v$, then it is not difficult to convince oneself that $(V^e)^+$ spans the whole self-adjoint part of V^e and that $(V^e)^+ \cap -(V^e)^+ = \{0\}$. The non-negative reals \mathbb{R}^+ share the same properties in \mathbb{C} . Then the inner product is a $*$ -map which takes $(V^e)^+$ into \mathbb{R}^+ . This motivates the following definition.

Consider the category \mathbf{HMod}_+^* of pairs (V, V^+) where V is a $*$ -module and V^+ , the *positive cone* of V , is a subset consisting of self-adjoint elements of V which is closed under addition and multiplication by \mathbb{R}^+ . We also require that $V^+ \cap (-V^+) = \{0\}$ and that $V^+ - V^+ = V^{sa}$, the real subspace of self-adjoint elements of V . A morphism $T : (V, V^+) \rightarrow (W, W^+)$ in \mathbf{HMod}_+^* is a $*$ -module map $T : V \rightarrow W$ such that $T(V^+) \subseteq W^+$. Then there is a functor from \mathbf{HMod} to \mathbf{HMod}_+^* given on objects by $V \mapsto V^e = \bar{V} \otimes V$ and on morphisms by $T \mapsto \bar{T} \otimes T$, and we define the positive cone of V^e to be the \mathbb{R}^+ -span of vectors of the form $\bar{v} \otimes v$ for $v \in V$, as above. The pair $(\mathbb{C}, \mathbb{R}^+)$ is an object of \mathbf{HMod}_+^* , and then an inner product on V would be defined to be a morphism \langle , \rangle in \mathbf{HMod}_+^* from V^e to \mathbb{C} , with positive cones as above.

The necessity to define the positive cone and to include that as part of the data manifests itself in other situations where there may be more than one relevant notion of positivity in a single vector space. For example, in a $*$ -algebra A (in the usual sense, not in the sense presented here, although the discussion could be formulated in our setting as well) one can define the positive cone A^+ to be the \mathbb{R}^+ -span of elements of the form a^*a . Then we can say that a linear functional $f \in A^*$ is positive if $f(A^+) \subseteq \mathbb{R}^+$, and finally we can define another positive cone in A by declaring A^{++} to be the collection of self-adjoint elements of A which are mapped to \mathbb{R}^+ by all positive functionals (and then check that this forms a cone). Certainly $A^+ \subseteq A^{++}$, and there is equality if, for instance, A is a C^* -algebra, but in general these notions will not coincide.

Although this would give an even more structural approach to the study of inner products, for our purposes we don’t need to make things quite so abstract, so we confine this construction to this subsection. We caution also that this approach has not been carefully considered by the author, and some tweaking of the conditions above may be necessary to develop the approach rigorously.

8.3. Adjoints of linear operators. Now we show how to use our notion of inner products to frame adjoints of linear maps in our language. We begin with some motivational discussion on the usual formulation of adjoints. For this we restrict to finite-dimensional modules.

Remark 8.8. When V and W are finite-dimensional inner product spaces with (ordinary, sesquilinear) inner products $(,)_V$ and $(,)_W$, respectively, one defines the *adjoint* of a linear transformation $T : V \rightarrow W$ to be the unique linear map $T^\dagger : W \rightarrow V$ satisfying

$$(8.2) \quad (T^\dagger w, v)_V = (w, Tv)_W$$

for all $v \in V$ and $w \in W$. Then one shows that the following properties hold:

- (a) The map $\text{Hom}(V, W) \rightarrow \text{Hom}(W, V)$ given by $T \mapsto T^\dagger$ is antilinear.
- (b) Taking the adjoint again gives $(T^\dagger)^\dagger = T$ in $\text{Hom}(V, W)$.
- (c) If X is another finite-dimensional inner product space, then for any linear map $U : W \rightarrow X$, we have $(UT)^\dagger = T^\dagger U^\dagger$.

Rather than repeating the construction of the adjoint operator from scratch, we use the correspondence from Proposition 8.4 between sesquilinear inner products and inner products in our sense in order to transport the traditional notion of adjoint given above into our setting.

Notation 8.9. We make the following notational convention: for a Hermitian module (or just complex vector space) V with inner product $\langle \cdot, \cdot \rangle_V$ in the sense of Definition 8.1, we will always denote by $(\cdot, \cdot)_V$ the corresponding positive-definite, conjugate-symmetric, sesquilinear form given by

$$(v, w)_V = \langle \bar{v}, w \rangle_V$$

for $v, w \in V$, as discussed in Proposition 8.4. With this convention in mind, translating the defining property (8.2) of the adjoint transformation into our language gives

$$(8.3) \quad \langle \overline{T^\dagger w}, v \rangle_V = \langle \bar{w}, Tv \rangle_W$$

for $v \in V$ and $w \in W$.

In Proposition 8.10 we explore some properties of the adjoint map T^\dagger . Part (a) shows that the adjoint of a module map is a module map. Part (b), while somewhat technical in its statement, is really just formalizing the notion that the adjoint of a linear map is its conjugate transpose.

Proposition 8.10. *Let $V, W \in \mathbf{HMod}_f$ be finite-dimensional Hermitian modules, let $T : V \rightarrow W$ be a linear map, and let $T^\dagger : W \rightarrow V$ be the adjoint of T . Then:*

- (a) *T is a module map if and only if T^\dagger is a module map.*
- (b) *T^\dagger coincides with the composition*

$$W \xrightarrow{\sigma_W^{-1}} \overline{\overline{W}} \xrightarrow{\overline{\mu_W}} \overline{W^*} \xrightarrow{\overline{T^{\text{tr}}}} \overline{V^*} \xrightarrow{\overline{\mu_V^{-1}}} \overline{\overline{V}} \xrightarrow{\sigma_V} V$$

Proof. (a) Since $(T^\dagger)^\dagger = T$, we only need to do one direction of the proof. Assuming that T is a module map, we need to show that $T^\dagger(a \triangleright w) = a \triangleright (T^\dagger w)$ for $w \in W$ and $a \in H$. For any $v \in V$, we have

$$\begin{aligned} (T^\dagger(a \triangleright w), v)_V &= (a \triangleright w, Tv)_W \\ &= (w, a^* \triangleright (Tv))_W \\ &= (w, T(a^* \triangleright v))_W \\ &= (Tw, a^* \triangleright v)_V \\ &= (a \triangleright (Tw), v)_V, \end{aligned}$$

using (8.2), Proposition 8.4 (b), and the fact that T is a module map. Since this holds for all $v \in V$, we conclude that $T^\dagger(a \triangleright w) = a \triangleright (T^\dagger w)$, so T^\dagger is a module map.

- (b) Recall that c_{W^*} is the canonical antimodule map from W^* to its conjugate. For $w \in W$, we have

$$\overline{\mu_W}(\sigma_W^{-1}(w)) = c_{W^*}(\langle \bar{w}, - \rangle_W).$$

Applying $\overline{T^{\text{tr}}}$ to this gives

$$\begin{aligned}\overline{T^{\text{tr}}}(c_{W^*}(\langle \overline{w}, - \rangle_W)) &= c_{V^*}(\langle \overline{w}, T(-) \rangle_W) \\ &= c_{V^*}(\langle \overline{T^\dagger w}, - \rangle_V).\end{aligned}$$

Finally, applying $\sigma_V \overline{\mu_V^{-1}}$ now gives

$$\begin{aligned}\sigma_V \overline{\mu_V^{-1}}(c_{V^*}(\langle \overline{T^\dagger w}, - \rangle_V)) &= \sigma_V(\overline{T^\dagger w}) \\ &= T^\dagger w,\end{aligned}$$

which proves our claim. \square

We now turn to the module-theoretic properties of the map $T \mapsto T^\dagger$. Recall from Definition 2.12 that we have two actions of H on each of the linear spaces $\text{Hom}(V, W)$ and $\text{Hom}(W, V)$ which we called *left* and *right*. We know that taking the adjoint is an antilinear map $\text{Hom}(V, W) \rightarrow \text{Hom}(W, V)$, but *a priori* it is not clear whether this should be an antimodule map for the left or right actions on the two Hom-spaces.

It turns out that we need to use the left action for both of them. The point here is that since we are dealing with finite-dimensional modules, we have isomorphisms $\overline{V} \simeq V^*$ and $\text{Hom}_\ell(V, W) \simeq W \otimes V^*$ from Propositions 8.5 and 2.13, respectively. Combining these gives an isomorphism $\text{Hom}_\ell(V, W) \simeq W \otimes \overline{V}$, and similarly we have $\text{Hom}_\ell(W, V) \simeq V \otimes \overline{W}$. As in Remark 8.7, we can see that this is a result of our convention that inner products are maps defined on $\overline{V} \otimes V$; if we had chosen the opposite convention then we would have Hom_r instead of Hom_ℓ in the following:

Proposition 8.11. *Let $V, W \in \mathbf{HMod}_f$ be finite-dimensional Hermitian modules.*

- (a) *The map $\text{Hom}_\ell(V, W) \rightarrow \text{Hom}_\ell(W, V)$ given by $T \mapsto T^\dagger$ is an anti-isomorphism of modules.*
- (b) *The map $* : \overline{\text{Hom}_\ell(V, W)} \rightarrow \text{Hom}_\ell(W, V)$ associated to $T \mapsto T^\dagger$ by Proposition 4.6 is an isomorphism of modules.*

Proof. (a) We need to show that $(a \triangleright T)^\dagger = S(a)^* \triangleright T^\dagger$ for $T \in \text{Hom}_\ell(V, W)$ and $a \in H$, where the action of H on $\text{Hom}_\ell(V, W)$ is given by (2.5). We will do this by showing that

$$((a \triangleright T)^\dagger w, v)_V = ((S(a)^* \triangleright T^\dagger)w, v)_V$$

for all $v \in V$, $w \in W$, and $a \in H$. Then the result will follow from nondegeneracy of the sesquilinear form. For the sake of readability, we omit the \triangleright symbols for the H -actions on V and W in the following. Beginning with the right-hand side, we have

$$\begin{aligned}((S(a)^* \triangleright T^\dagger)w, v)_V &= (S(a_{(2)})^* T^\dagger S(a_{(1)})^* w, v)_V \\ &= (T^\dagger a_{(1)}^* w, S(a_{(2)})v)_V \\ &= (w, a_{(1)} T S(a_{(2)})v)_W \\ &= (w, (a \triangleright T)v)_W \\ &= ((a \triangleright T)^\dagger w, v)_V,\end{aligned}$$

where we used the property (8.2) of the adjoint, as well as the invariance of $(,)_V$ and $(,)_W$ under the H -action described in part (b) of Proposition 8.4.

(b) Follows immediately from (a) together with Proposition 4.6. \square

Remark 8.12. Although the notation is suggestive, the map $*$: $\overline{\text{Hom}_\ell(V, W)} \rightarrow \text{Hom}_\ell(W, V)$ is not a $*$ -structure in the sense of Definition 6.1 because it is not of the form $*$: $\overline{Y} \rightarrow Y$ for a module Y except when $V = W$. We address this case now. Since we have not yet shown that $\text{End}_\ell(V)$ is an H -module algebra, we take care of this detail first.

Proposition 8.13. *For $V \in \mathbf{HMod}$, the algebra $\text{End}_\ell(V)$ is an algebra in \mathbf{HMod} .*

Proof. We need to show that the conditions (5.1) hold. For $x, y \in \text{End}_\ell(V)$ and $a \in H$ we have

$$\begin{aligned} (a_{(1)} \triangleright x)(a_{(2)} \triangleright y) &= a_{(1)}xS(a_{(2)})a_{(3)}yS(a_{(4)}) \\ &= a_{(1)}x\varepsilon(a_{(2)})yS(a_{(3)}) \\ &= a_{(1)}xyS(\varepsilon(a_{(2)})a_{(3)}) \\ &= a_{(1)}xyS(a_{(2)}) \\ &= a \triangleright (xy), \end{aligned}$$

so multiplication in $\text{End}_\ell(V)$ is a module map. For the unit map, we write $1 = \text{id}_V$; we have

$$a \triangleright 1 = a_{(1)}1S(a_{(2)}) = \varepsilon(a)1,$$

so the unit is also a module map, which concludes the proof. \square

Proposition 8.14. *Let $V \in \mathbf{HMod}_f$ be a finite-dimensional Hermitian module. Then the map $*$: $\overline{\text{End}_\ell(V)} \rightarrow \text{End}_\ell(V)$ from Proposition 8.11 makes $\text{End}_\ell(V)$ into a $*$ -algebra in \mathbf{HMod} .*

Proof. We know already from Proposition 8.13 that $\text{End}_\ell(V)$ is an H -module algebra. We know also from Proposition 8.11 that $T \mapsto T^\dagger$ is an anti-isomorphism of modules, and according to the properties (b) and (c) listed in Remark 8.8, we see that this map is actually an involutive antimodule-algebra morphism. Thus $*$ is a $*$ -structure by Proposition 7.2. \square

Remark 8.15. As in the discussion preceding Proposition 8.11, note that $\text{End}_\ell(V) \simeq V \otimes \overline{V}$, which we denoted by eV in §6.2. We note that the $*$ -structure on $\text{End}_\ell(V)$ from Proposition 8.14 corresponds through this isomorphism with the $*$ -structure on eV defined in Proposition 6.10.

8.4. Inner products, $*$ -structures, and bilinear forms. Let $V \in \mathbf{HMod}_f$. In this subsection we explore the relationship between inner products, $*$ -structures, and bilinear forms on V .

By definition, a $*$ -structure on V is an isomorphism $*$: $\overline{V} \rightarrow V$ with a certain extra property (namely involutivity). If V is also a Hermitian module, then we can define a bilinear form $h : V \otimes V \rightarrow \mathbb{C}$ by $h(\overline{v}^*, w) = \langle \overline{v}, w \rangle$. In other words, we define the bilinear form so that the following diagram commutes:

$$(8.4) \quad \begin{array}{ccc} \overline{V} \otimes V & \xrightarrow{* \otimes \text{id}} & V \otimes V \\ & \searrow \langle \cdot, \cdot \rangle & \downarrow h \\ & & \mathbb{C} \end{array}$$

Since $*$ is an isomorphism and since the inner product is nondegenerate, the bilinear form h will be nondegenerate as well. Note that h is a morphism in \mathbf{HMod} .

Remark 8.16. A $*$ -structure on V gives an isomorphism $\overline{V} \simeq V$. An inner product on V gives an isomorphism $\overline{V} \simeq V^*$ via $\overline{v} \mapsto \langle \overline{v}, - \rangle$; see Proposition 8.5. Similarly, a nondegenerate bilinear form $h : V \otimes V \rightarrow \mathbb{C}$ gives an isomorphism $V \simeq V^*$ via $v \mapsto h(v, -)$.

We saw above that having a $*$ -structure and an inner product allowed us to obtain a bilinear form. This corresponds to composing the corresponding isomorphisms $V \simeq \overline{V}$ and $\overline{V} \simeq V^*$ to obtain the isomorphism $V \simeq V^*$ associated to the bilinear form. We would now like to ask if there is a “two-out-of-three” type result, i.e. whether having any two of a bilinear form, an inner product, and a $*$ -structure, allows us to obtain the third one.

In the discussion preceding this remark, we got the bilinear form for free. But we will see that in fact this bilinear form necessarily satisfies some conditions, so it is not the case that, for example, an arbitrary bilinear form plus an inner product will give a $*$ -structure. Similarly, we cannot expect an arbitrary bilinear form plus a $*$ -structure to give an inner product. Thus a completely general two-out-of-three result is not possible. But if we restrict the bilinear forms we consider, we do obtain some results.

Proposition 8.17. *Let $V \in \mathbf{HMod}_f$ and let $* : \overline{V} \rightarrow V$, $\langle , \rangle : \overline{V} \otimes V \rightarrow \mathbb{C}$, and $h : V \otimes V \rightarrow \mathbb{C}$ be module maps such that the diagram (8.4) commutes (we do not assume $*$ to be a $*$ -structure nor \langle , \rangle to be an inner product). Then*

- (a) *If \langle , \rangle is an inner product on V , then $*$ is a $*$ -structure on V if and only if $h(\overline{w}^*, \overline{v}^*) = \overline{h(v, w)}$ for all $v, w \in V$.*
- (b) *If $*$ is a $*$ -structure on V , then \langle , \rangle is an inner product on V if and only if h satisfies the conditions $h(\overline{w}^*, \overline{v}^*) = \overline{h(v, w)}$ for $v, w \in V$, $h(\overline{v}^*, v) \geq 0$ for $v \in V$, and $h(\overline{v}^*, v) = 0$ only if $v = 0$.*

Proof. Consider the following diagram:

$$(8.5) \quad \begin{array}{ccccc} \overline{V} \otimes \overline{V} & \xrightarrow{\rho_{\overline{V}V}} & \overline{V} \otimes \overline{V} & \xrightarrow{\text{id} \otimes \sigma_V} & \overline{V} \otimes V \\ & \searrow * \otimes \text{id} & \downarrow \text{id} \otimes * & \nearrow \text{id} \otimes * & \downarrow * \otimes \text{id} \\ \overline{V} \otimes V & \xrightarrow{\rho_{VV}} & \overline{V} \otimes V & \xrightarrow{* \otimes *} & V \otimes V \\ \downarrow \langle , \rangle & \nearrow \overline{h} & & \searrow h & \downarrow \langle , \rangle \\ \mathbb{C} & \xrightarrow{\gamma} & \mathbb{C} & & \mathbb{C} \end{array}$$

Certain parts of this diagram commute automatically. The triangle on the right-hand side commutes since it is exactly (8.4), and the triangle on the left-hand side is the complex conjugate of that on the right, so it commutes as well. The trapezoid on the upper left commutes by Proposition 4.8. The small interior triangle commutes trivially.

This leaves the triangle on the upper right, the pentagon, and the large rectangle. The triangle on the upper right commutes if and only if $*$ is a $*$ -structure. The

pentagon commutes if and only if $h(\overline{w}^*, \overline{v}^*) = \overline{h(v, w)}$ for all $v, w \in V$. Noting that the composition across the top of (8.5) is exactly the $*$ -structure on $\overline{V} \otimes V$ described in Proposition 6.10, we see that the large rectangle commutes if and only if h is a $*$ -map.

If $\langle \cdot, \cdot \rangle$ is an inner product, it is a $*$ -map, and hence the large rectangle commutes. Then the upper right triangle commutes if and only if the pentagon commutes, i.e. $*$ is a $*$ -structure if and only if $h(\overline{w}^*, \overline{v}^*) = \overline{h(v, w)}$ for all $v, w \in V$. This proves (a).

On the other hand, if $*$ is a $*$ -structure, then the upper right triangle commutes. Thus the large rectangle commutes if and only if the pentagon commutes, i.e. $\langle \cdot, \cdot \rangle$ is a $*$ -map if and only if $h(\overline{w}^*, \overline{v}^*) = \overline{h(v, w)}$ for all $v, w \in V$. From (8.4) we have $h(\overline{v}^*, w) = \langle \overline{v}, w \rangle$, so we see that $\langle \cdot, \cdot \rangle$ satisfies the appropriate positivity requirement if and only if $h(\overline{v}^*, v) \geq 0$ for $v \in V$ and $h(\overline{v}^*, v) = 0$ only if $v = 0$. This proves (b). \square

Remark 8.18. We now explain the meaning of the condition $h(\overline{w}^*, \overline{v}^*) = \overline{h(v, w)}$. We said in the proof of Proposition 8.17 that this condition is equivalent to commutativity of the pentagon in the diagram (8.5). If $*$ is a $*$ -structure on V , then the composition $(* \otimes *) \circ \rho_{VV}$ (i.e. the top of the pentagon) is the $*$ -structure on $V \otimes V$ described in 6.12 (for $n = 2$). Then the statement that the pentagon commutes means exactly that $h : V \otimes V \rightarrow \mathbb{C}$ is a $*$ -map. We could also formulate the positivity criterion $h(\overline{v}^*, v) \geq 0$ in terms of positive cones, as discussed in §8.2.

9. *-STRUCTURES AND BRAIDINGS

In this section we examine the interaction between $*$ -structures, R-matrices, and braidings. We show that the complex conjugate of a braiding on the module category of a Hopf $*$ -algebra is again a braiding. When the braiding comes from an R-matrix, we show how conditions on the R-matrix translate into relations between the braiding and its conjugate, and relations between the braiding and the $*$ -structure on modules. In particular, when the R-matrix is *real* (see Definition 9.10), then the braiding of V with itself is a module map for any $*$ -module V .

9.1. Braidings and their complex conjugates. In this subsection we define a braiding and show that the complex conjugate of a braiding on a category of H -modules gives another braiding. Although the notion of braiding makes sense for any monoidal category, we define it here only for **HMod** (or for a sub-monoidal category of **HMod**). This simplifies our presentation because we already understand the monoidal structure well: it is the tensor product of vector spaces over \mathbb{C} . Thus we can and do suppress the associativity isomorphisms $U \otimes (V \otimes W) \simeq (U \otimes V) \otimes W$.

Definition 9.1. Let \mathcal{C} be a subcategory of **HMod** containing \mathbb{C} and such that $V \otimes W \in \mathcal{C}$ for all $V, W \in \mathcal{C}$ (a *sub-monoidal category*). A *braiding* on \mathcal{C} is a collection of isomorphisms $\psi_{VW} : V \otimes W \rightarrow W \otimes V$ (in \mathcal{C}) for all $V, W \in \mathcal{C}$ satisfying the conditions

$$(9.1) \quad \begin{aligned} \psi_{U, V \otimes W} &= (\text{id}_V \otimes \psi_{UW})(\psi_{UV} \otimes \text{id}_W), & \psi_{U \otimes V, W} &= (\psi_{U, W} \otimes \text{id}_V)(\text{id}_U \otimes \psi_{VW}), \end{aligned}$$

and such that for any modules U, V, W, X and any morphisms $f : U \rightarrow W$ and $g : V \rightarrow X$ in \mathcal{C} , the following diagram commutes:

$$(9.2) \quad \begin{array}{ccc} U \otimes V & \xrightarrow{\psi_{UV}} & V \otimes U \\ f \otimes g \downarrow & & \downarrow g \otimes f \\ W \otimes X & \xrightarrow{\psi_{WX}} & X \otimes W \end{array}$$

We refer to the property (9.2) as *naturality* of the braiding. We refer to the relations (9.1) as the *hexagon axioms*; the reason for this is that the relevant diagrams would each form a hexagon if we did not suppress the associators.

Remark 9.2. A fancier way to define a braiding is as follows. One can form the Cartesian product category $\mathcal{C} \times \mathcal{C}$. Then the tensor product in \mathcal{C} defines a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, which takes a pair of objects (U, V) to their tensor product $U \otimes V$, and a pair of morphisms (f, g) to the tensor product morphism $f \otimes g$. The fact that the tensor product is associative means that there is a natural isomorphism between the functors $\otimes \circ (\otimes \times \text{id})$ and $\otimes \circ (\text{id} \times \otimes)$ from $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ to \mathcal{C} . Similarly, there is a functor \otimes^{op} which takes the pairs (U, V) and (f, g) to $V \otimes U$ and $g \otimes f$, respectively. Then a braiding can be defined as a natural isomorphism between the functors \otimes and \otimes^{op} .

It follows from the axioms that the braidings $\psi_{\mathbb{C}V}$ and $\psi_{V\mathbb{C}}$ are compatible with the canonical identifications of $\mathbb{C} \otimes V$ and $V \otimes \mathbb{C}$ with V . We refer to §2 of [JS93] for further details on braidings in general.

Our task now is to show that complex conjugation of a braiding is again a braiding. While somewhat tedious, this is essentially straightforward. For the rest of §9.1, assume that \mathcal{C} is subcategory of **HMod** which is equipped with a braiding $\psi = (\psi_{VW})_{V, W \in \mathcal{C}}$ as in Definition 9.1. Suppose furthermore that \mathcal{C} is closed under complex conjugation, i.e. that $\overline{V} \in \mathcal{C}$ for all $V \in \mathcal{C}$ and \overline{f} is a morphism in \mathcal{C} for all morphisms f in \mathcal{C} . Finally, assume that the isomorphisms $\rho_{VW} : \overline{V} \otimes \overline{W} \rightarrow \overline{W} \otimes \overline{V}$ and $\sigma_V : \overline{\overline{V}} \rightarrow V$ from Proposition 4.8 and Lemma 3.11, respectively, along with their inverses, are in \mathcal{C} as well.

We now define a new family of isomorphisms obtained from the original braiding ψ by conjugation. As this is somewhat technical, we make the definition in two stages. First, for each $X, Y \in \mathcal{C}$, we define an isomorphism $\xi_{XY} : \overline{X} \otimes \overline{Y} \rightarrow \overline{Y} \otimes \overline{X}$ via the composition

$$(9.3) \quad \xi_{XY} : \overline{X} \otimes \overline{Y} \xrightarrow{\rho_{YX}^{-1}} \overline{Y} \otimes \overline{X} \xrightarrow{\overline{\psi_{XY}}} \overline{X} \otimes \overline{Y} \xrightarrow{\rho_{XY}} \overline{Y} \otimes \overline{X}.$$

Note that our assumptions ensure that ξ_{XY} is a morphism in \mathcal{C} .

We have now provided an isomorphism ξ_{XY} between the modules $\overline{X} \otimes \overline{Y}$ and $\overline{Y} \otimes \overline{X}$ for any X and Y . In order to define the conjugated braiding, we use the fact that every module is naturally isomorphic to the conjugate of some module (namely to the conjugate of its conjugate):

Definition 9.3. For any $X, Y \in \mathcal{C}$ we define an isomorphism $\overline{\psi}_{XY}$ (not to be confused with $\overline{\psi_{XY}}$) from $X \otimes Y$ to $Y \otimes X$ via the composition

$$(9.4) \quad \overline{\psi}_{XY} : X \otimes Y \xrightarrow{\sigma_X^{-1} \otimes \sigma_Y^{-1}} \overline{\overline{X}} \otimes \overline{\overline{Y}} \xrightarrow{\xi_{\overline{\overline{X}}\overline{\overline{Y}}}} \overline{\overline{Y}} \otimes \overline{\overline{X}} \xrightarrow{\sigma_Y \otimes \sigma_X} Y \otimes X.$$

We refer to the family $\overline{\psi} = (\overline{\psi}_{XY})_{X, Y \in \mathcal{C}}$ as the *conjugate braiding* to ψ .

As the name suggests, the family $\overline{\psi}$ is also a braiding on \mathcal{C} . Verifying the details is a little tedious. We begin by proving the relevant properties of the maps ξ_{XY} before using them to establish the properties of $\overline{\psi}$. First we show that the ξ_{XY} 's are natural with respect to module maps.

Lemma 9.4. *Let $U, V, W, X \in \mathcal{C}$ and let $f : U \rightarrow W$ and $g : V \rightarrow X$ be morphisms in \mathcal{C} . Then the following diagram commutes:*

$$(9.5) \quad \begin{array}{ccc} \overline{U} \otimes \overline{V} & \xrightarrow{\xi_{UV}} & \overline{V} \otimes \overline{U} \\ \overline{f} \otimes \overline{g} \downarrow & & \downarrow \overline{g} \otimes \overline{f} \\ \overline{W} \otimes \overline{X} & \xrightarrow{\xi_{WX}} & \overline{X} \otimes \overline{W} \end{array}$$

Proof. This follows from (9.2) together with (4.2) (used twice). \square

The harder part of showing that $\overline{\psi}$ is a braiding is checking that the hexagon axioms (9.1) hold. We now prove a technical lemma, which will assist us in checking these relations.

Lemma 9.5. *Let $U, V, W \in \mathcal{C}$. The following diagram commutes:*

$$(9.6) \quad \begin{array}{ccc} \overline{U} \otimes \overline{V \otimes W} & \xrightarrow{\xi_{U, V \otimes W}} & \overline{V \otimes W} \otimes \overline{U} \\ \text{id}_{\overline{U}} \otimes \rho_{VW} \downarrow & & \downarrow \rho_{VW} \otimes \text{id}_{\overline{U}} \\ \overline{U} \otimes \overline{W} \otimes \overline{V} & \xrightarrow{\xi_{UW} \otimes \text{id}_{\overline{V}}} \overline{W} \otimes \overline{U} \otimes \overline{V} \xrightarrow{\text{id}_{\overline{W}} \otimes \xi_{UV}} \overline{W} \otimes \overline{V} \otimes \overline{U} \end{array}$$

Proof. Commutativity of (9.6) follows from commutativity of the diagram (9.7). Indeed, expanding all of the ξ 's in (9.6) according to the definition gives the outer edges of (9.7), so if (9.7) commutes then (9.6) will as well.

(9.7)

$$\begin{array}{ccccccc} \overline{U} \otimes \overline{V \otimes W} & \xrightarrow{\rho^{-1}} & \overline{V \otimes W} \otimes \overline{U} & \xrightarrow{\overline{\psi}} & \overline{U \otimes V} \otimes \overline{W} & \xrightarrow{\rho} & \overline{V \otimes W} \otimes \overline{U} \xrightarrow{\rho \otimes \text{id}} \overline{W} \otimes \overline{V} \otimes \overline{U} \\ \downarrow \text{id} \otimes \rho & & \downarrow \rho & \swarrow \text{id} \otimes \overline{\psi} \quad \searrow \overline{\psi} \otimes \text{id} & & & \downarrow \rho \\ & & \overline{V \otimes U} \otimes \overline{W} & & & & \\ & & \downarrow \rho & & & & \\ \overline{U} \otimes \overline{W} \otimes \overline{V} & \xrightarrow{\rho^{-1} \otimes \text{id}} \overline{W} \otimes \overline{U} \otimes \overline{V} & \xrightarrow{\overline{\psi} \otimes \text{id}} \overline{U} \otimes \overline{W} \otimes \overline{V} & \xrightarrow{\rho \otimes \text{id}} \overline{W} \otimes \overline{U} \otimes \overline{V} & \xrightarrow{\text{id} \otimes \rho^{-1}} \overline{W} \otimes \overline{V} \otimes \overline{U} & \xrightarrow{\text{id} \otimes \overline{\psi}} \overline{W} \otimes \overline{U} \otimes \overline{V} \end{array}$$

We have omitted the subscripts labeling the objects on the ρ 's and ψ 's for legibility. This should cause no confusion because there is only one choice of subscripts in each case that matches the domain and codomain of each arrow.

It is left to prove that the diagram above commutes. We begin at the left-hand side and move to the right. The rectangle commutes by (4.3). The trapezoid commutes by (4.2). The upper triangle commutes since ψ is a braiding. The lower triangle commutes by (4.3). The quadrilateral commutes by (4.2). Finally, the triangle on the right-hand side commutes by (4.3). This completes the proof of the lemma. \square

We are now ready to prove our first main result in this section:

Proposition 9.6. *The conjugate braiding $\bar{\psi}$ is a braiding on \mathcal{C} .*

Proof. First we check the naturality property (9.2). Let $U, V, W, X \in \mathcal{C}$ and let $f : U \rightarrow W$ and $g : V \rightarrow X$ be morphisms in \mathcal{C} . Then we claim that the following diagram commutes (where we have removed most of the subscripts on the maps for legibility):

$$(9.8) \quad \begin{array}{ccccccc} U \otimes V & \xrightarrow{\sigma^{-1} \otimes \sigma^{-1}} & \bar{U} \otimes \bar{V} & \xrightarrow{\xi_{\bar{U}\bar{V}}} & \bar{V} \otimes \bar{U} & \xrightarrow{\sigma \otimes \sigma} & V \otimes U \\ f \otimes g \downarrow & & \bar{f} \otimes \bar{g} \downarrow & & \downarrow \bar{g} \otimes \bar{f} & & \downarrow g \otimes f \\ W \otimes X & \xrightarrow{\sigma^{-1} \otimes \sigma^{-1}} & \bar{W} \otimes \bar{X} & \xrightarrow{\xi_{\bar{W}\bar{X}}} & \bar{X} \otimes \bar{W} & \xrightarrow{\sigma \otimes \sigma} & X \otimes W \end{array}$$

Each of the two outer squares is a tensor product of two diagrams of the form (3.4), so they commute by Lemma 3.11. The central square commutes by Lemma 9.4 applied to the modules $\bar{U}, \bar{V}, \bar{W}, \bar{X}$ and the morphisms \bar{f} and \bar{g} . Noting that the compositions across the top and bottom of (9.8) are exactly $\bar{\psi}_{UV}$ and $\bar{\psi}_{WX}$, respectively, this completes the proof of naturality.

For the hexagon axioms, we will verify only the equality

$$(9.9) \quad \bar{\psi}_{X,Y \otimes Z} = (\text{id}_Y \otimes \bar{\psi}_{XZ})(\bar{\psi}_{XY} \otimes \text{id}_Z);$$

the other one is similar. Consider the following diagram:

$$\begin{array}{ccccc} X \otimes Y \otimes Z & \xrightarrow{\sigma^{-1} \otimes \sigma^{-1}} & \bar{X} \otimes \bar{Y} \otimes \bar{Z} & \xrightarrow{\xi} & \bar{Y} \otimes \bar{Z} \otimes \bar{X} & \xrightarrow{\sigma \otimes \sigma} & Y \otimes Z \otimes X \\ \downarrow \sigma^{-1} \otimes \sigma^{-1} \otimes \text{id} & & \downarrow \text{id} \otimes \bar{\rho} & & \downarrow \bar{\rho} \otimes \text{id} & & \nearrow \sigma \otimes \sigma \otimes \sigma \\ & & \bar{X} \otimes \bar{Z} \otimes \bar{Y} & \xrightarrow{\xi} & \bar{Z} \otimes \bar{Y} \otimes \bar{X} & & \\ & & \downarrow \text{id} \otimes \rho & & \downarrow \rho \otimes \text{id} & & \\ & & \bar{X} \otimes \bar{Y} \otimes \bar{Z} & & \bar{Y} \otimes \bar{Z} \otimes \bar{X} & & \\ \downarrow \xi \otimes \text{id} & & \downarrow \xi \otimes \text{id} & \nearrow \text{id} \otimes \xi & \downarrow \sigma \otimes \text{id} \otimes \text{id} & & \downarrow \text{id} \otimes \sigma \otimes \sigma \\ \bar{X} \otimes \bar{Y} \otimes Z & \xrightarrow{\text{id} \otimes \text{id} \otimes \sigma^{-1}} & \bar{X} \otimes \bar{Y} \otimes \bar{Z} & & \bar{Y} \otimes \bar{Z} \otimes \bar{X} & & \\ \downarrow \sigma \otimes \sigma \otimes \text{id} & & \downarrow \sigma \otimes \text{id} \otimes \text{id} & \nearrow \sigma \otimes \xi & \downarrow \sigma \otimes \text{id} \otimes \text{id} & & \\ Y \otimes X \otimes Z & \xrightarrow{\text{id} \otimes \sigma^{-1} \otimes \sigma} & Y \otimes \bar{X} \otimes \bar{Z} & \xrightarrow{\text{id} \otimes \xi} & Y \otimes \bar{Z} \otimes \bar{X} & & \end{array}$$

Although this looks rather unpleasant, it is mostly harmless. Almost all of the pieces commute tautologically. The only pieces that are not obvious are the square in the top middle part of the diagram, and the pentagon immediately below that square. The square commutes by Lemma 9.4, and the pentagon commutes by Lemma 9.5.

To complete the proof, note that the left-hand side of (9.9) is exactly the composition along the top row of the diagram, while the right-hand side of (9.9) is the composition down the left side, across the bottom, and up to the top right corner. \square

9.2. Quasitriangular Hopf *-algebras. Quasitriangular Hopf algebras are important because their module categories are automatically equipped with a braiding. Our goal in §9.2 is to understand what the conjugate braiding $\overline{\psi}$ looks like when H is a quasitriangular Hopf *-algebra. We begin by recalling the definition and properties of quasitriangular Hopf algebras.

Recall that a Hopf algebra H is *quasitriangular* [KS97, Chapter 8] if there is an invertible element $\mathcal{R} \in H \otimes H$ such that

$$(9.10) \quad \mathcal{R}\Delta(a)\mathcal{R}^{-1} = \Delta^{op}(a) \stackrel{\text{def}}{=} \tau \circ \Delta(a),$$

and

$$(9.11) \quad (\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$

where τ is the tensor flip. The element \mathcal{R} is called a *universal R-matrix* for H . In (9.11) we are using the so-called *leg-numbering notation*: if $\mathcal{R} = \sum_j x_j \otimes y_j$, then $\mathcal{R}_{12} = \sum_j x_j \otimes y_j \otimes 1$, $\mathcal{R}_{13} = \sum_j x_j \otimes 1 \otimes y_j$, and $\mathcal{R}_{23} = \sum_j 1 \otimes x_j \otimes y_j$. We also denote $\mathcal{R}_{21} = \tau(\mathcal{R}) = \sum_j y_j \otimes x_j$.

The important point here is that if H is quasitriangular, then \mathbf{HMod} acquires a braiding from the action of the R-matrix. Before describing the braiding, we record here for later use some of the consequences of quasitriangularity. The proofs of these statements can be found in §8.1 of [KS97].

Proposition 9.7. *Suppose that H is a quasitriangular Hopf algebra with universal R-matrix \mathcal{R} . Then the following hold:*

$$(9.12) \quad \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

$$(9.13) \quad (\varepsilon \otimes \text{id})(\mathcal{R}) = (\text{id} \otimes \varepsilon)(\mathcal{R}) = 1$$

$$(9.14) \quad (S \otimes \text{id})(\mathcal{R}) = \mathcal{R}^{-1}, \quad (\text{id} \otimes S)(\mathcal{R}^{-1}) = \mathcal{R}, \quad (S \otimes S)(\mathcal{R}) = \mathcal{R}.$$

Furthermore, the element $u = m(S \otimes \text{id})(\mathcal{R}_{21})$ is invertible in H with inverse $u^{-1} = m(\text{id} \otimes S^2)(\mathcal{R}_{21})$ (here m denotes the multiplication map of H), and the element $uS(u) = S(u)u$ is central in H . The antipode of H is invertible, and we have

$$(9.15) \quad S^2(a) = uau^{-1}, \quad S^{-1}(a) = u^{-1}S(a)u$$

for all $a \in H$.

The relation (9.12) is called the *quantum Yang-Baxter equation*.

If U, V are any two H -modules, then \mathcal{R} acts in $U \otimes V$ coordinate-wise. Then we have the following result [KS97, §8.1.2]:

Proposition 9.8. *Suppose that H is a quasitriangular Hopf algebra with universal R-matrix \mathcal{R} . For any $U, V \in \mathbf{HMod}$, define*

$$\psi_{UV} = \tau \circ \mathcal{R} : U \otimes V \rightarrow V \otimes U.$$

Then $\psi = (\psi_{UV})_{U, V \in \mathbf{HMod}}$ is a braiding on \mathbf{HMod} .

Now that the prerequisites are in place, we can begin to explore the effect of conjugation on the braiding coming from the R-matrix.

Notation 9.9. We extend the *-structure of H to a *-structure on $H \otimes H$ coordinate-wise. For a simple tensor $a \otimes b \in H \otimes H$, we have $(a \otimes b)^* = a^* \otimes b^*$. In particular,

for $\mathcal{R} = \sum_j x_j \otimes y_j$ we have

$$\mathcal{R}^* = \sum_j x_j^* \otimes y_j^*.$$

The following definition is inspired by [KS97] Definition 2 in §10.1.1:

Definition 9.10. Suppose that H is a quasitriangular Hopf $*$ -algebra with universal R-matrix \mathcal{R} . We say that \mathcal{R} is *real* if $\mathcal{R}^* = \mathcal{R}_{21}$. We say that \mathcal{R} is *inverse real* if $\mathcal{R}^* = \mathcal{R}^{-1}$.

When the R-matrix is either real or inverse real, we can draw some conclusions about the conjugated braiding:

Proposition 9.11. *Suppose that H is a quasitriangular Hopf $*$ -algebra with universal R-matrix \mathcal{R} . Let ψ be the braiding on \mathbf{HMod} as in Proposition 9.8*

- (a) *If \mathcal{R} is real, then $\bar{\psi} = \psi$, i.e. for any $U, V \in \mathbf{HMod}$ we have $\bar{\psi}_{UV} = \psi_{UV}$.*
- (b) *If \mathcal{R} is inverse real, then $\bar{\psi} = \psi^{-1}$, i.e. for any $U, V \in \mathbf{HMod}$ we have $\bar{\psi}_{UV} = \psi_{VU}^{-1}$.*

Proof. Let us write $\mathcal{R} = x_j \otimes y_j$ with implied summation. First we compute the maps ξ_{VW} that we defined in (9.3). For $v \in V$, $w \in W$ we have

$$\begin{aligned} \xi_{VW}(\bar{v} \otimes \bar{w}) &= \rho_{VW} \left(\overline{\psi_{WV}(w \otimes v)} \right) \\ &= \rho_{VW} \left(\overline{\tau \circ \mathcal{R}(w \otimes v)} \right) \\ &= \rho_{VW} \left(\overline{(y_j \triangleright v) \otimes (x_j \triangleright w)} \right) \\ &= \overline{(x_j \triangleright w)} \otimes \overline{(y_j \triangleright v)} \\ &= (S(x_j)^* \triangleright \bar{w}) \otimes (S(y_j)^* \triangleright \bar{v}) \\ &= [(S \otimes S)(\mathcal{R})^*](\bar{w} \otimes \bar{v}) \\ &= \mathcal{R}^*(\bar{w} \otimes \bar{v}) \\ &= (\tau \circ \mathcal{R}_{21}^*)(\bar{v} \otimes \bar{w}). \end{aligned}$$

In other words, we have shown that $\xi_{VW} = \tau \circ \mathcal{R}_{21}^*$.

When \mathcal{R} is real, then we have

$$\xi_{VW} = \tau \circ \mathcal{R}_{21}^* = \tau \circ \mathcal{R} = \psi_{\bar{V}\bar{W}}.$$

Applying naturality of the braiding ψ to the morphisms σ_V and σ_W (and suppressing subscripts for readability), we have

$$\bar{\psi}_{VW} = (\sigma \otimes \sigma) \xi_{\bar{V}\bar{W}} (\sigma^{-1} \otimes \sigma^{-1}) = (\sigma \otimes \sigma) \psi_{\bar{V}\bar{W}} (\sigma^{-1} \otimes \sigma^{-1}) = \psi_{VW}.$$

This establishes (a).

When \mathcal{R} is inverse real, then our initial computation gives

$$\xi_{VW} = \tau \circ \mathcal{R}_{21}^* = \tau \circ \mathcal{R}_{21}^{-1} = \mathcal{R}^{-1} \circ \tau = \psi_{\bar{W}\bar{V}}^{-1};$$

as above, using naturality of the braiding ψ , we conclude that $\bar{\psi}_{VW} = \psi_{\bar{W}\bar{V}}^{-1}$. This establishes (b). \square

9.3. Braidings and *-structures. For this subsection we will assume that H is a quasitriangular Hopf *-algebra, so that \mathbf{HMod} is braided. As above, we denote the braiding coming from the R-matrix by ψ . Recall from Proposition 6.12 that if V is a *-module, then $V \otimes V$ is also a *-module, with *-structure given by $(\overline{u \otimes v})^* = \overline{v}^* \otimes \overline{u}^*$. It therefore makes sense to ask: under what circumstances is the braiding map ψ_{VV} a morphism of *-modules? In other words, when does the following diagram commute:

$$(9.16) \quad \begin{array}{ccc} \overline{V \otimes V} & \xrightarrow{*} & V \otimes V \\ \overline{\psi_{VV}} \downarrow & & \downarrow \psi_{VV} \\ \overline{V \otimes V} & \xrightarrow{*} & V \otimes V \end{array}$$

Note that on the left-hand side of (9.16) we really mean the complex conjugate of the map ψ_{VV} , not the conjugated braiding $\overline{\psi}_{VV}$ (which wouldn't make sense anyway).

Proposition 9.12. *Let $V \in \mathbf{HMod}$ be a *-module, and give $V \otimes V$ the *-structure described in Proposition 6.12. If the R-matrix \mathcal{R} of H is real, then the diagram (9.16) commutes, so ψ_{VV} is a morphism of *-modules.*

Proof. Again, let us denote the universal R-matrix by $\mathcal{R} = x_j \otimes y_j$, with implied summation. Let $u, v \in V$. On the one hand, we have

$$\begin{aligned} \psi_{VV}((\overline{u \otimes v})^*) &= \psi_{VV}(\overline{v}^* \otimes \overline{u}^*) \\ &= (\tau \circ \mathcal{R})(\overline{v}^* \otimes \overline{u}^*) \\ &= (y_j \triangleright \overline{u}^*) \otimes (x_j \triangleright \overline{v}^*) \\ &= \mathcal{R}_{21}(\overline{u}^* \otimes \overline{v}^*). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (\overline{\psi_{VV}(u \otimes v)})^* &= (\overline{\psi_{VV}(u \otimes v)})^* \\ &= (\overline{(\tau \circ \mathcal{R})(u \otimes v)})^* \\ &= (\overline{(y_j \triangleright v) \otimes (x_j \triangleright u)})^* \\ &= \overline{(x_j \triangleright u)}^* \otimes \overline{(y_j \triangleright v)}^* \\ &= (S(x_j)^* \triangleright \overline{u}^*) \otimes (S(y_j)^* \triangleright \overline{v}^*) \\ &= (S \otimes S)(\mathcal{R})^*(\overline{u}^* \otimes \overline{v}^*) \\ &= \mathcal{R}^*(\overline{u}^* \otimes \overline{v}^*). \end{aligned}$$

If \mathcal{R} is real, then $\mathcal{R}^* = \mathcal{R}_{21}$ by definition, so the two expressions agree, and thus (9.16) commutes. Hence ψ_{VV} is a *-map. \square

REFERENCES

- [BK01] Bojko Bakalov and Alexander Kirillov Jr., *Lectures on tensor categories and modular functors*, University Lecture Series, vol. 21, American Mathematical Society, Providence, RI, 2001. MR1797619 (2002d:18003)
- [BBT07] Margaret Beattie, Daniel Bulacu, and Blas Torrecillas, *Radford's S^4 formula for co-Frobenius Hopf algebras*, J. Algebra **307** (2007), no. 1, 330–342, DOI 10.1016/j.jalgebra.2006.06.004. MR2278058 (2007h:16043)

- [BB09] Margaret Beattie and Daniel Bulacu, *On the antipode of a co-Frobenius (co)quasitriangular Hopf algebra*, Comm. Algebra **37** (2009), no. 9, 2981–2993, DOI 10.1080/00927870802502647. MR2554186 (2010i:16055)
- [CP95] Vyjayanthi Chari and Andrew Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1995. Corrected reprint of the 1994 original. MR1358358 (96h:17014)
- [FY92] Peter Freyd and David N. Yetter, *Coherence theorems via knot theory*, J. Pure Appl. Algebra **78** (1992), no. 1, 49–76, DOI 10.1016/0022-4049(92)90018-B. MR1154897 (93d:18013)
- [JS93] André Joyal and Ross Street, *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78, DOI 10.1006/aima.1993.1055. MR1250465 (94m:18008)
- [KS97] Anatoli Klimyk and Konrad Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997. MR1492989 (99f:17017)
- [Mon93] Susan Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993. MR1243637 (94i:16019)
- [Rad76] David E. Radford, *The order of the antipode of a finite dimensional Hopf algebra is finite*, Amer. J. Math. **98** (1976), no. 2, 333–355. MR0407069 (53 #10852)
- [Tak71] Mitsuhiro Takeuchi, *Free Hopf algebras generated by coalgebras*, J. Math. Soc. Japan **23** (1971), 561–582. MR0292876 (45 #1958)
- [Vin94] È. B. Vinberg (ed.), *Lie groups and Lie algebras, III*, Encyclopaedia of Mathematical Sciences, vol. 41, Springer-Verlag, Berlin, 1994. Structure of Lie groups and Lie algebras; A translation of *Current problems in mathematics. Fundamental directions. Vol. 41* (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990 [MR1056485 (91b:22001)]; Translation by V. Minachin [V. V. Minakhin]; Translation edited by A. L. Onishchik and È. B. Vinberg. MR1349140 (96d:22001)

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